The Little Black Book of Math Art

Searching for Patterns

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Introduction

Do a search for the meaning of mathematics and you are certain to find a great many definitions. Mathematics is the study of such things as quantity, structure, space, change, numbers, measurement, combination, interrelation, shape, and arrangement.

This book looks at mathematics as a search for patterns, patterns arising from arithmetic, geometry, number theory, and algebra. Some of the patterns are displayed as single pictures, but others are revealed by displaying a sequences of pictures, each picture derived from the previous one based on some rule. There is a pattern to the creation of a pattern.

All of the examples presented in this book were developed using software written by the author. You can learn more about the software, and download free copies of it, by visiting Syzygy Shareware at https://tcbretl.weebly.com.

Mod Arithmetic

Shown here are mod 5 arithmetic tables for addition and multiplication. Each number in the first row is added to or multiplied by each number in the first column. The answer is then divided by 5, and the remainder is displayed in the appropriate cell.

5 becomes 0	5 = 0 mod 5
6 becomes 1	6 = 1 mod 5
7 becomes 2	7 = 2 mod 5
8 becomes 3	8 = 3 mod 5
9 becomes 4	9 = 4 mod 5
10 becomes 0	10 = 0 mod 5

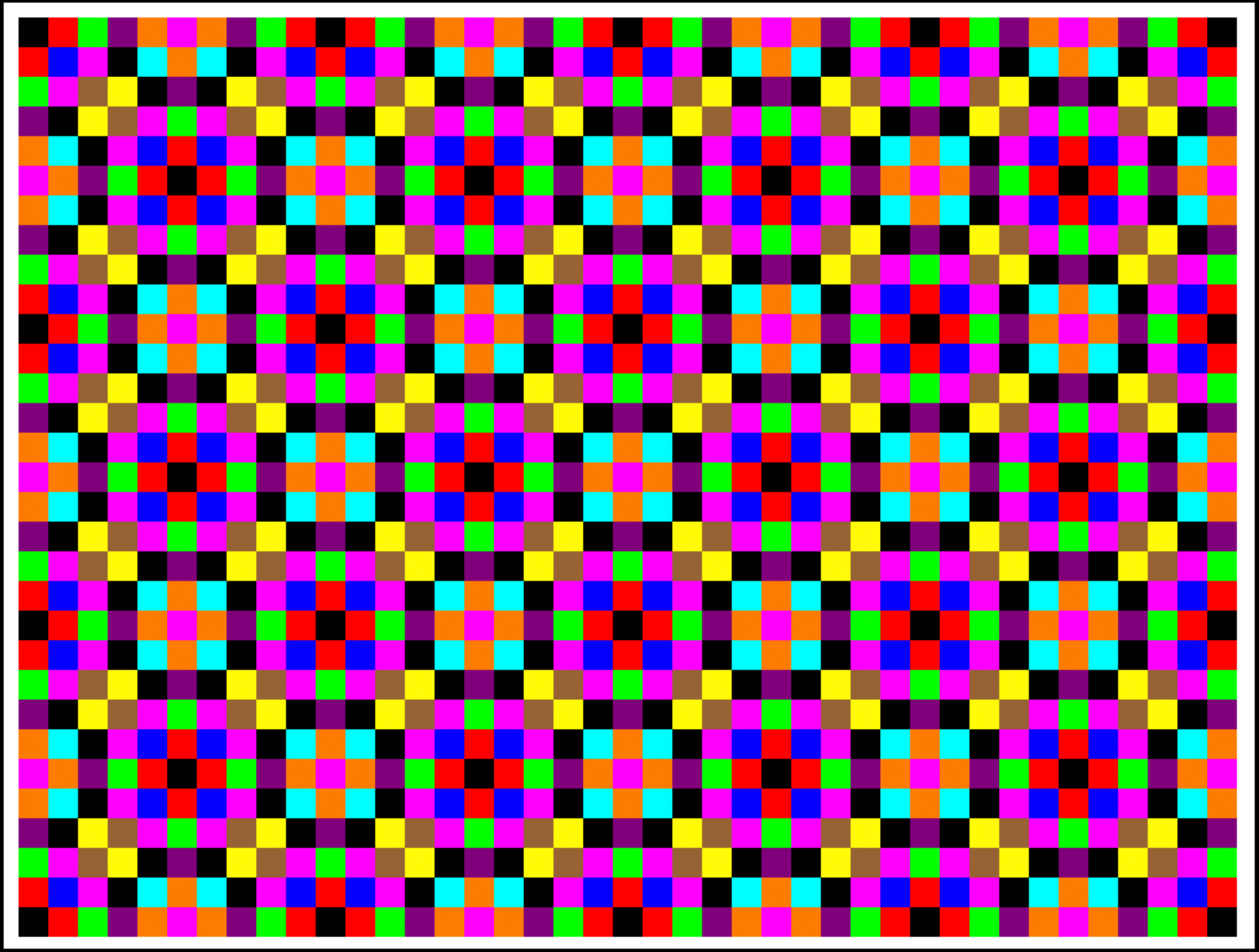
To reveal the patterns more clearly and dramatically, each number is represented by a different color.

Other rules for combining the numbers can also be used. You could both add and multiply the numbers, for example, or square them before you add them.

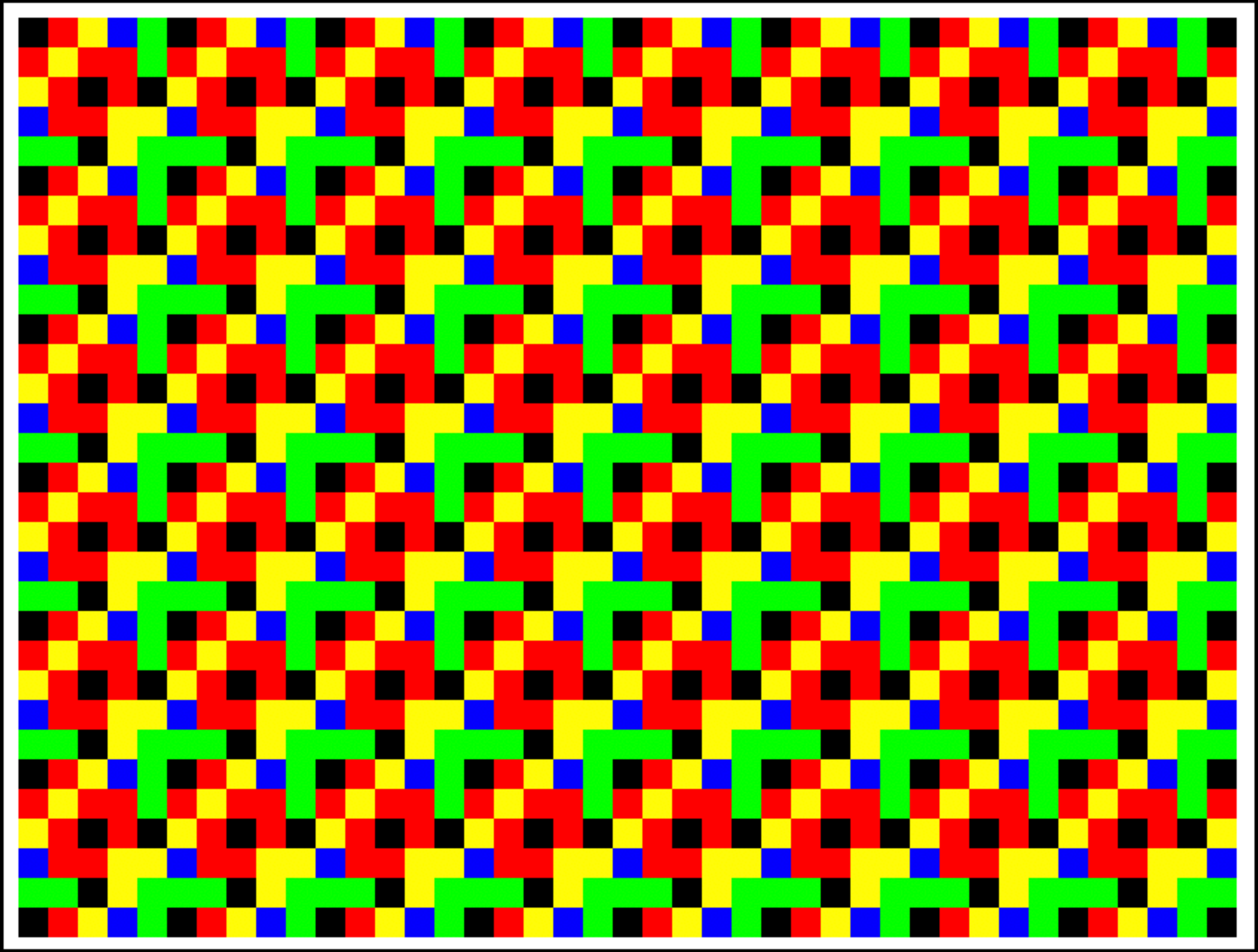
+	0	1	2	3	4	5	6	7	8	9
0	0	1	2	3	4	0	1	2	3	4
1	1	2	3	4	0	1	2	3	4	0
2	2	3	4	0	1	2	3	4	0	1
3	3	4	0	1	2	3	4	0	1	2
4	4	0	1	2	3	4	0	1	2	3
5	0	1	2	3	4	0	1	2	3	4
6	1	2	3	4	0	1	2	3	4	0
7	2	3	4	0	1	2	3	4	0	1
8	3	4	0	1	2	3	4	0	1	2
9	4	0	1	2	3	4	0	1	2	3

X	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	0	1	2	3	4
2	0	2	4	1	3	0	2	4	1	3
3	0	3	1	4	2	0	3	1	4	2
4	0	4	3	2	1	0	4	3	2	1
5	0	0	0	0	0	0	0	0	0	0
6	0	1	2	3	4	0	1	2	3	4
7	0	2	4	1	3	0	2	4	1	3
8	0	3	1	4	2	0	3	1	4	2
9	0	4	3	2	1	0	4	3	2	1

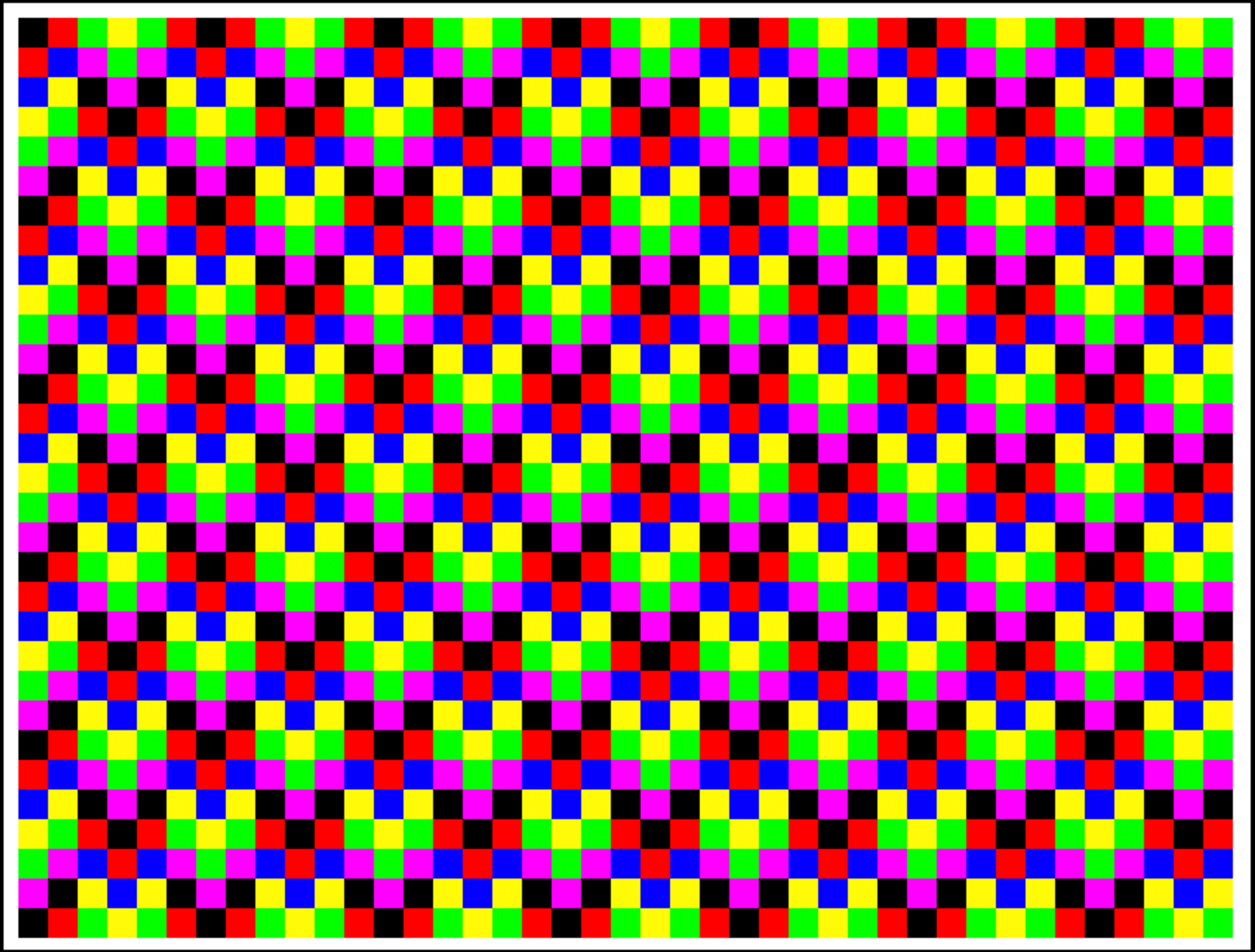
$x^2 + y^2 \bmod 10$



$x^3 + y^3 + xy \bmod 5$



$x^2 + y^3 \bmod 6$



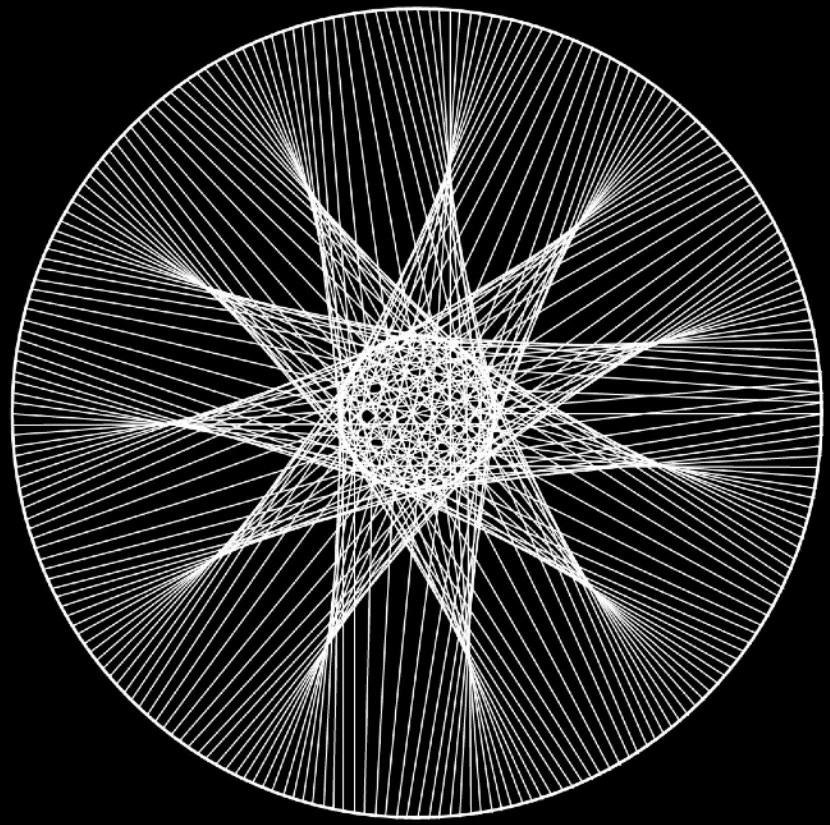
Planetary Motion

Suppose you could look down on the solar system and observe the motions of the planets and their moons from afar. Consider, for example, the motions of the earth and Jupiter.

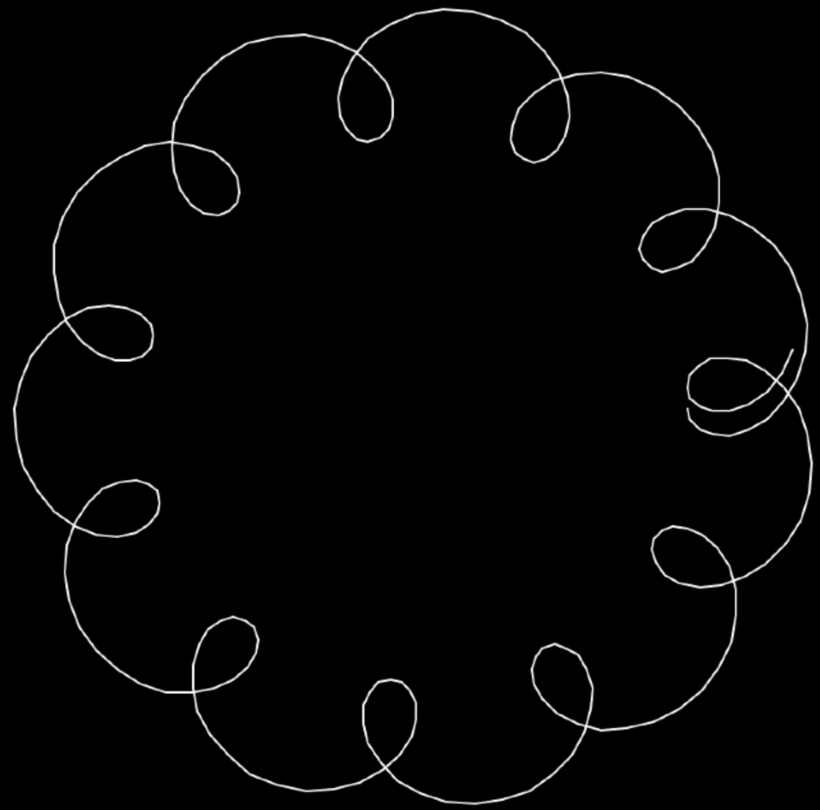
If you positioned yourself such that sun appeared stationary, the top diagram on the right shows what it would look like if you periodically drew a line from the earth to Jupiter over a period of approximately 12 years. The earth goes around the sun 12 times while Jupiter completes just one orbit, creating a string art pattern with 11 points (one point for each time the earth “passes” Jupiter).

But if you positioned yourself such that the earth appeared stationary, the bottom diagram shows how Jupiter’s motion would appear over that same period of time. Because of the earth’s yearly motion, Jupiter follows a loopy pattern as it completes just one orbit. Because Jupiter takes slightly less than 12 years to make one revolution, however, the pattern shifts slightly for the next time around.

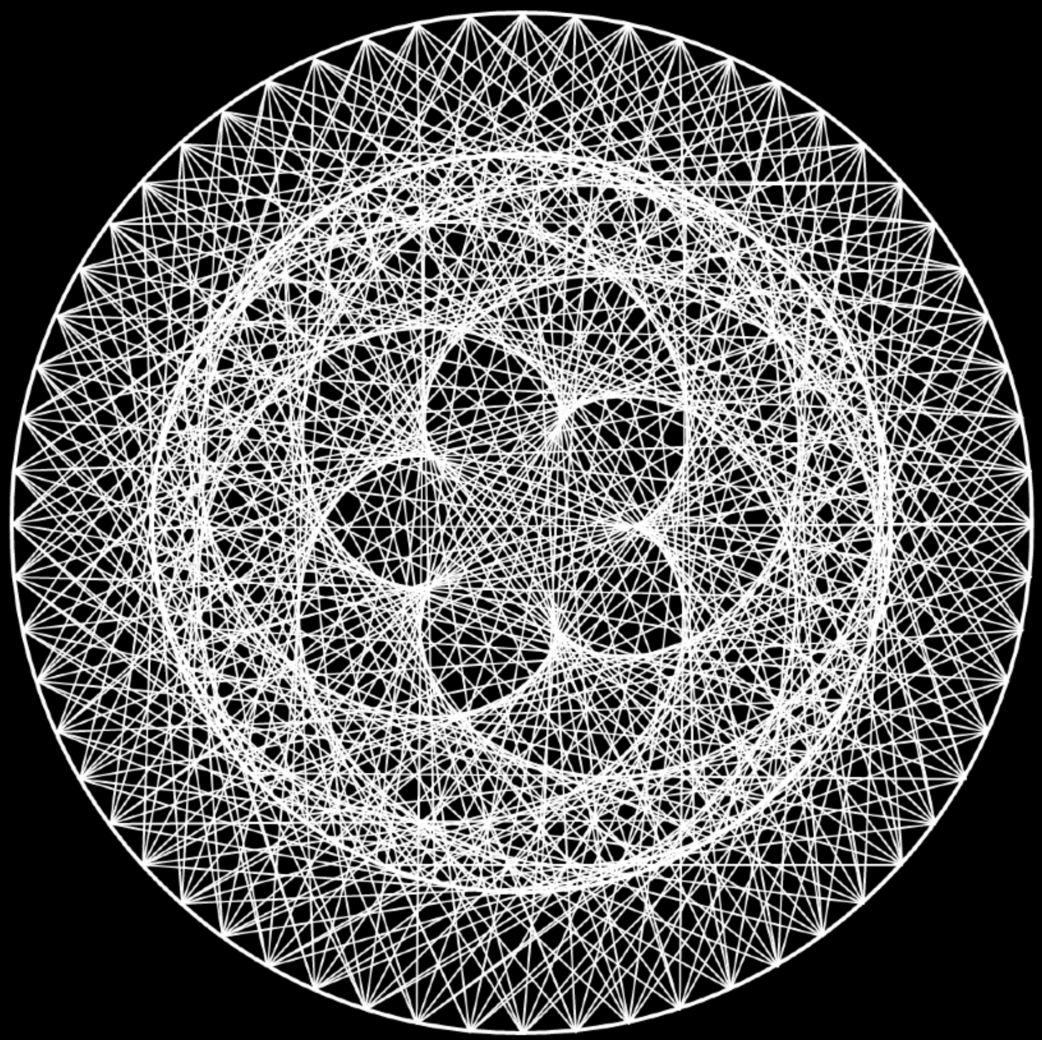
Sun centered



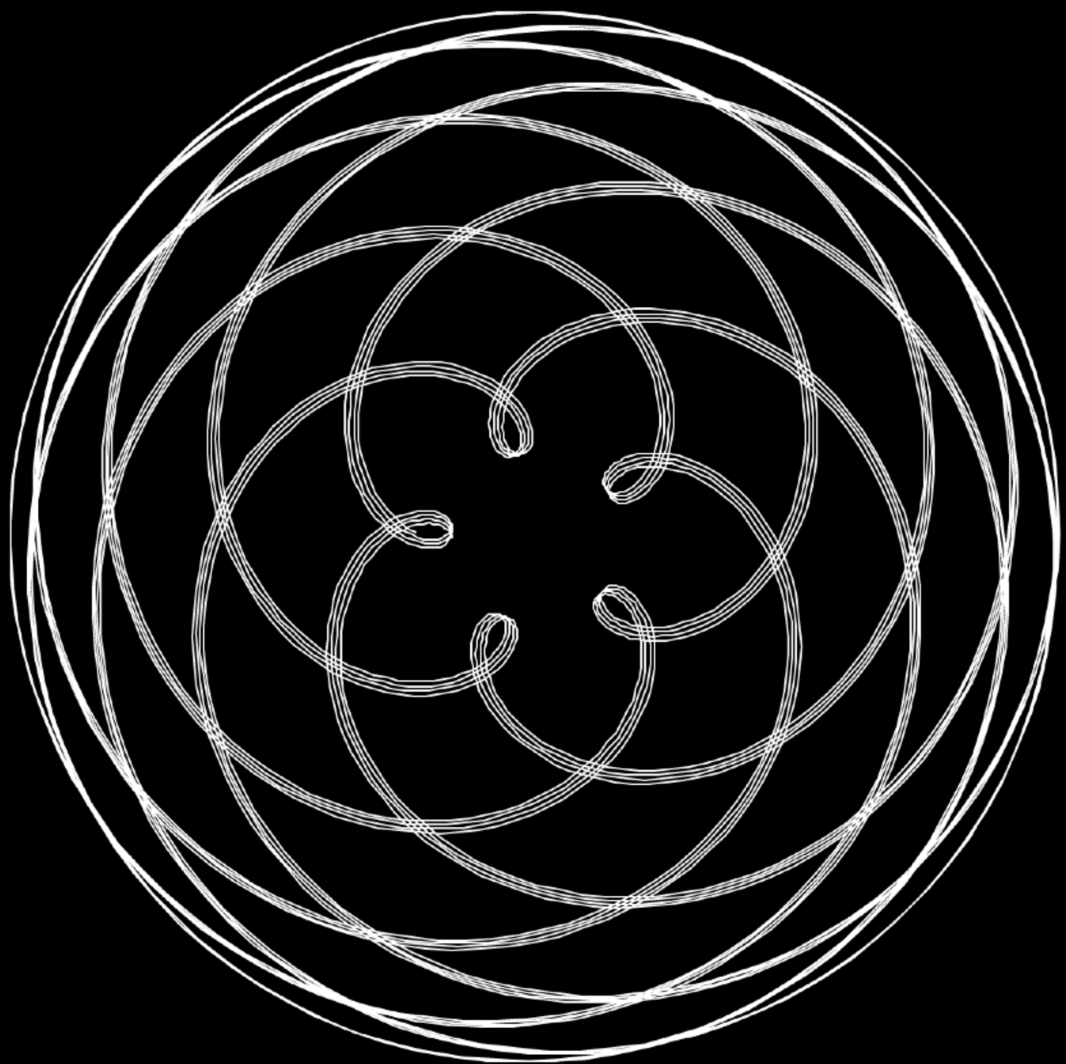
Earth centered



Sun with Earth and Venus

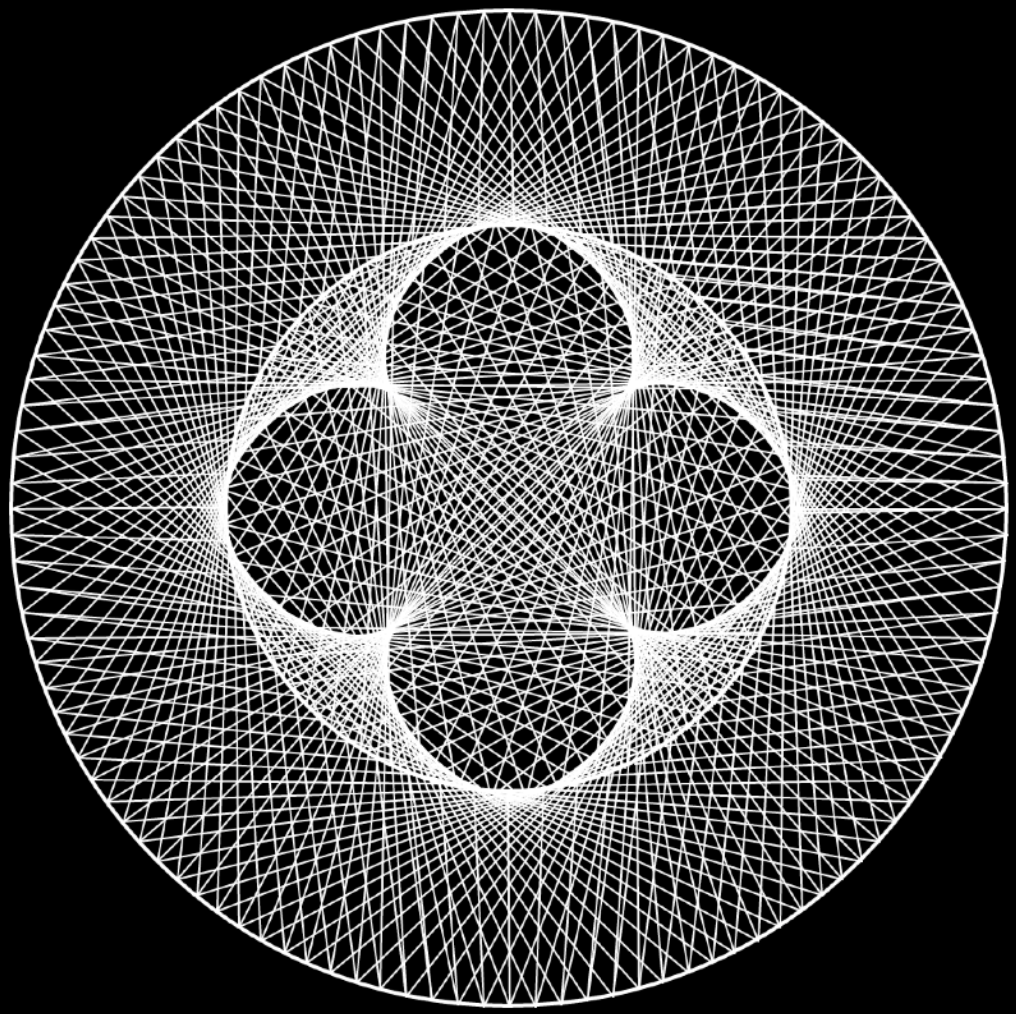


Sun centered (8 years)

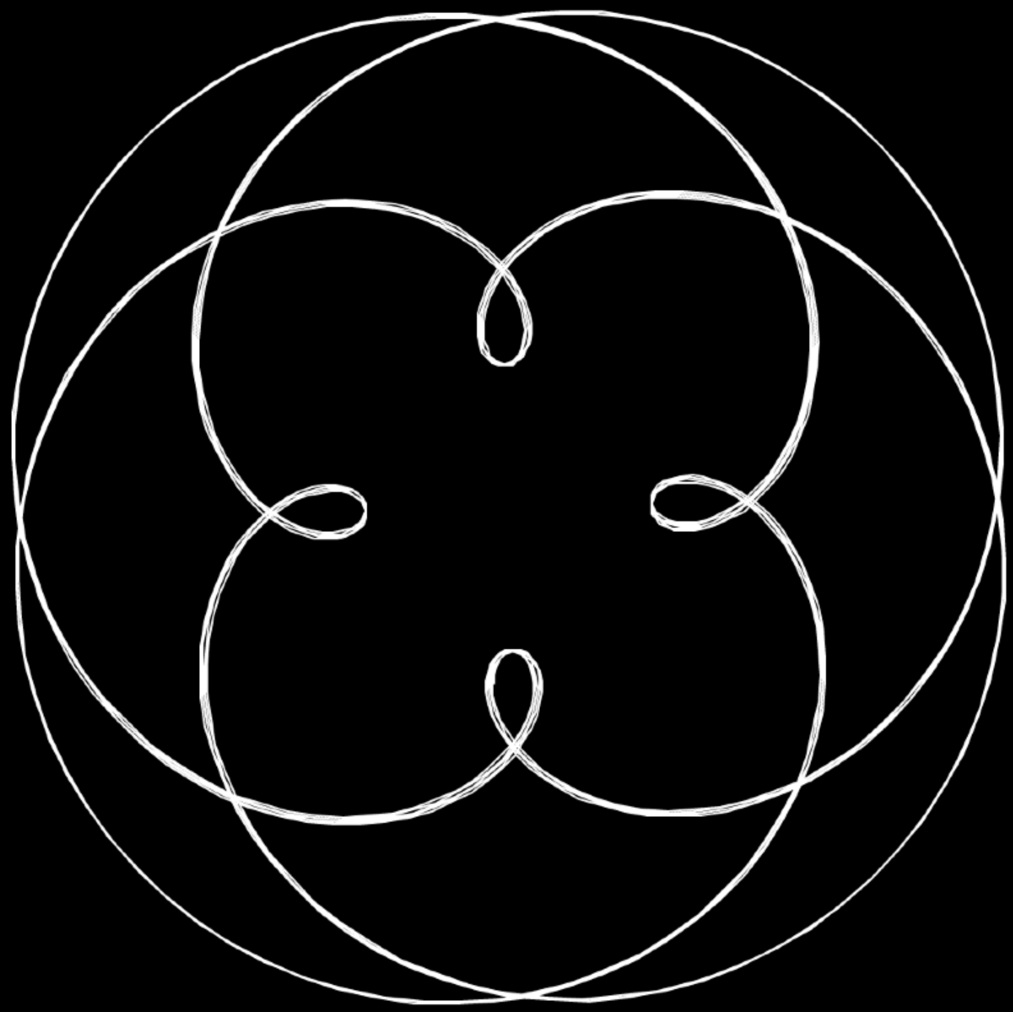


Earth centered (32 years)

Jupiter with moons Ganymede and Callisto



Jupiter centered (50 days)



Ganymede centered (50 days)

Prime Numbers

Suppose you listed all of the whole numbers between 1 and 20100 in a grid with 134 rows and 150 columns arranged like this:

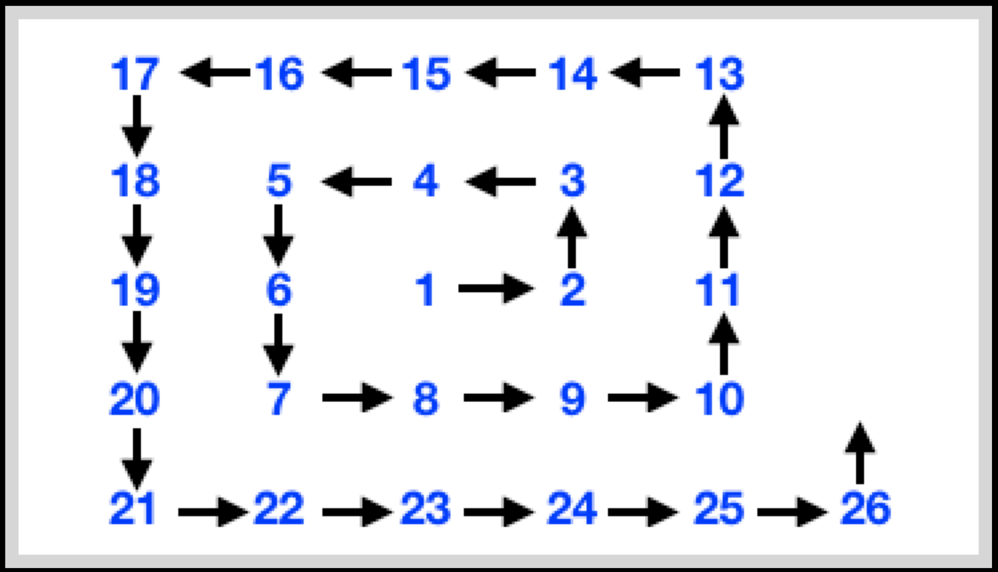
1	2	3	⋯	149	150
151	152	153	⋯	199	300
301	302	303	⋯	449	450

Replace the prime numbers with a plotted point and just erase all of the non-primes. The result is shown here. All of the primes between 1 and 20,000 are included.

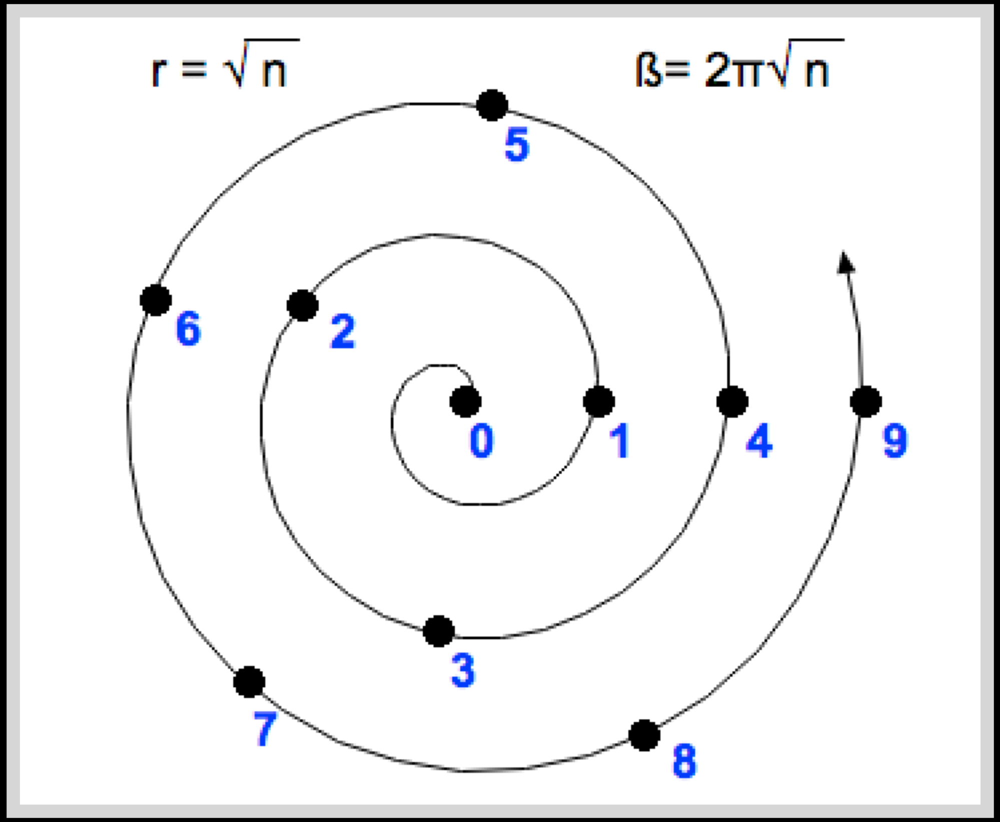
Notice the gaps between some of the columns. The points colored red represent consecutive primes which differ by 6, the first ending in 3 and the second ending in 9. Because each row starts with n = 1 mod 150, these prime pairs line up nicely directly below each other.

Plotting along Spirals

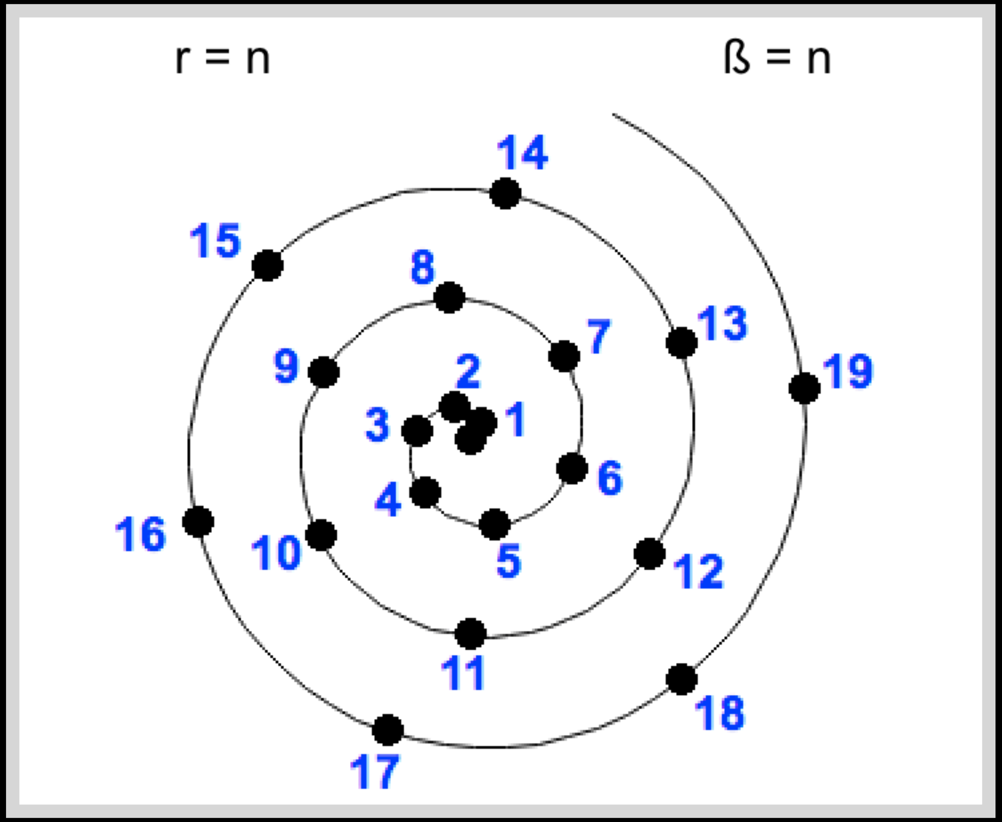
More surprising patterns are revealed if you start by evenly spacing the numbers along a spiral, either in a rectangular fashion as shown to the right, or based on polar coordinates as shown below. For the Archimedian spirals, both r and ß depend on the number n being plotted.



Square (Ulam) Spiral



Archimedian Spiral 1



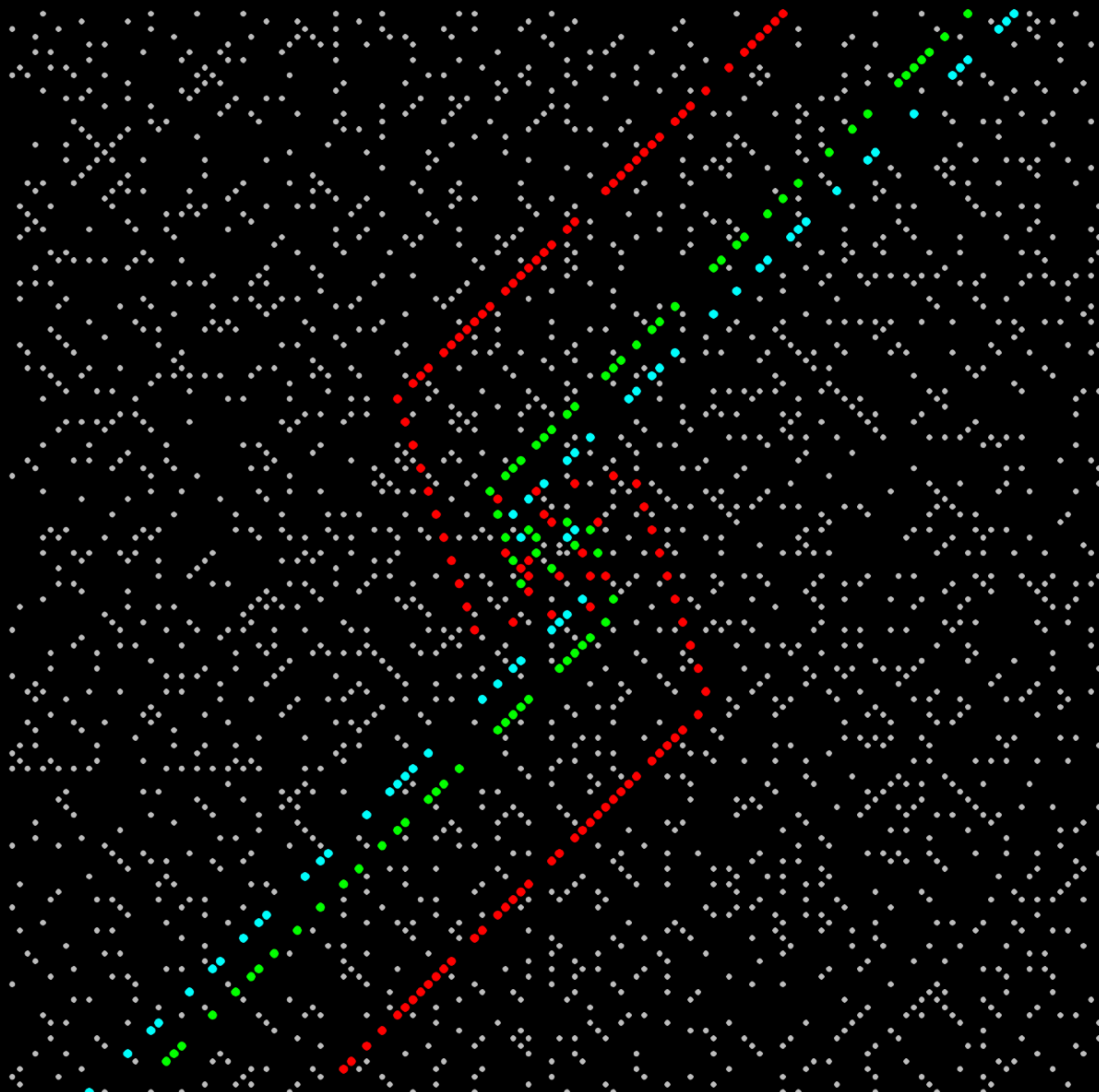
Archimedian Spiral 2

Primes plotted along the Ulam Spiral

Primes generated by $n^2 + n + 11$ (11, 13, 17, 23 ...)

Primes generated by $n^2 + n + 17$ (11, 13, 17, 23 ...)

Primes generated by $n^2 + n + 17$ (11, 13, 17, 41 ...)

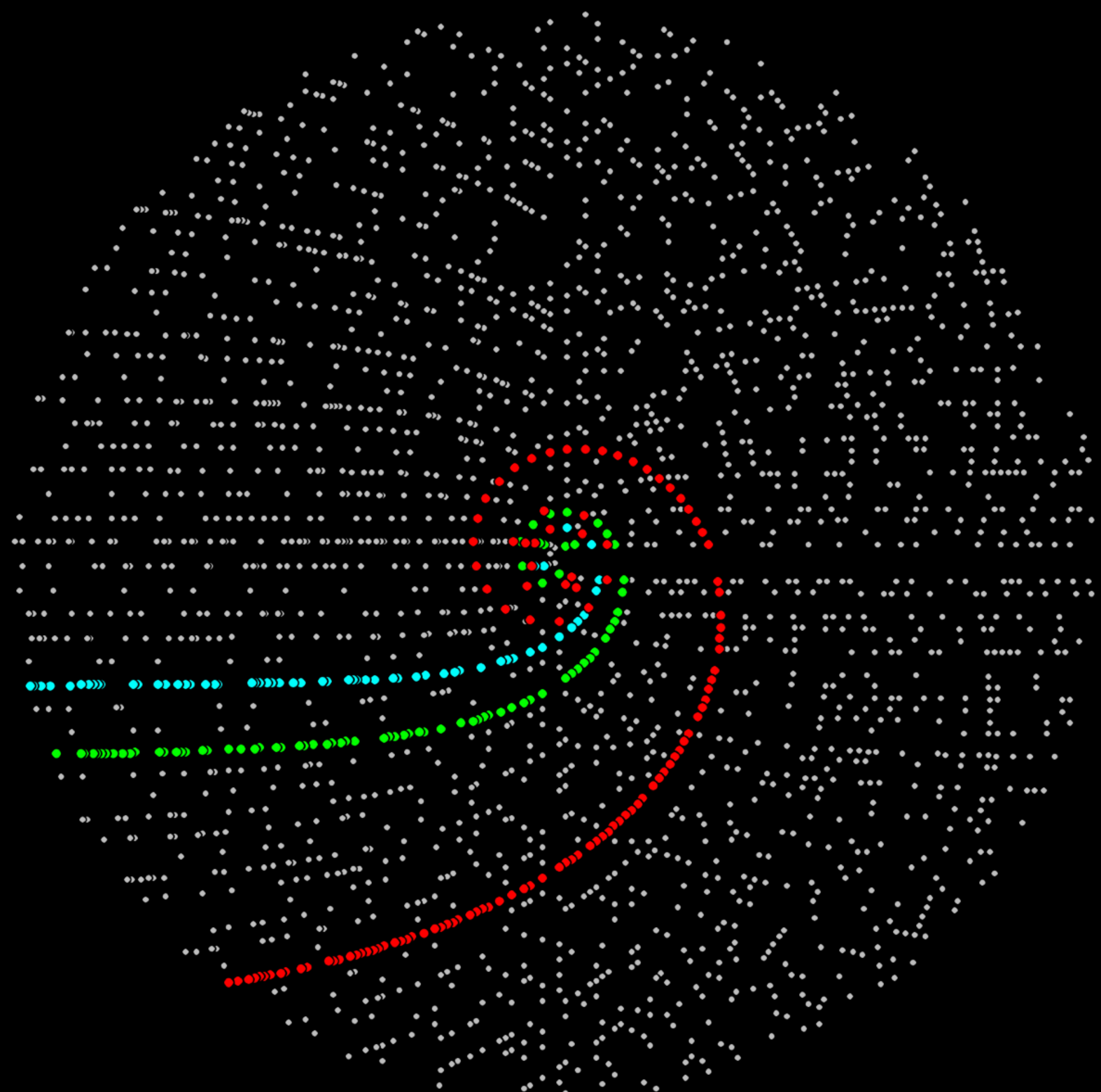


Primes plotted along Archimedian Spiral 1

Primes generated
by $n^2 + n + 11$
(11, 13, 17, 23 ...)

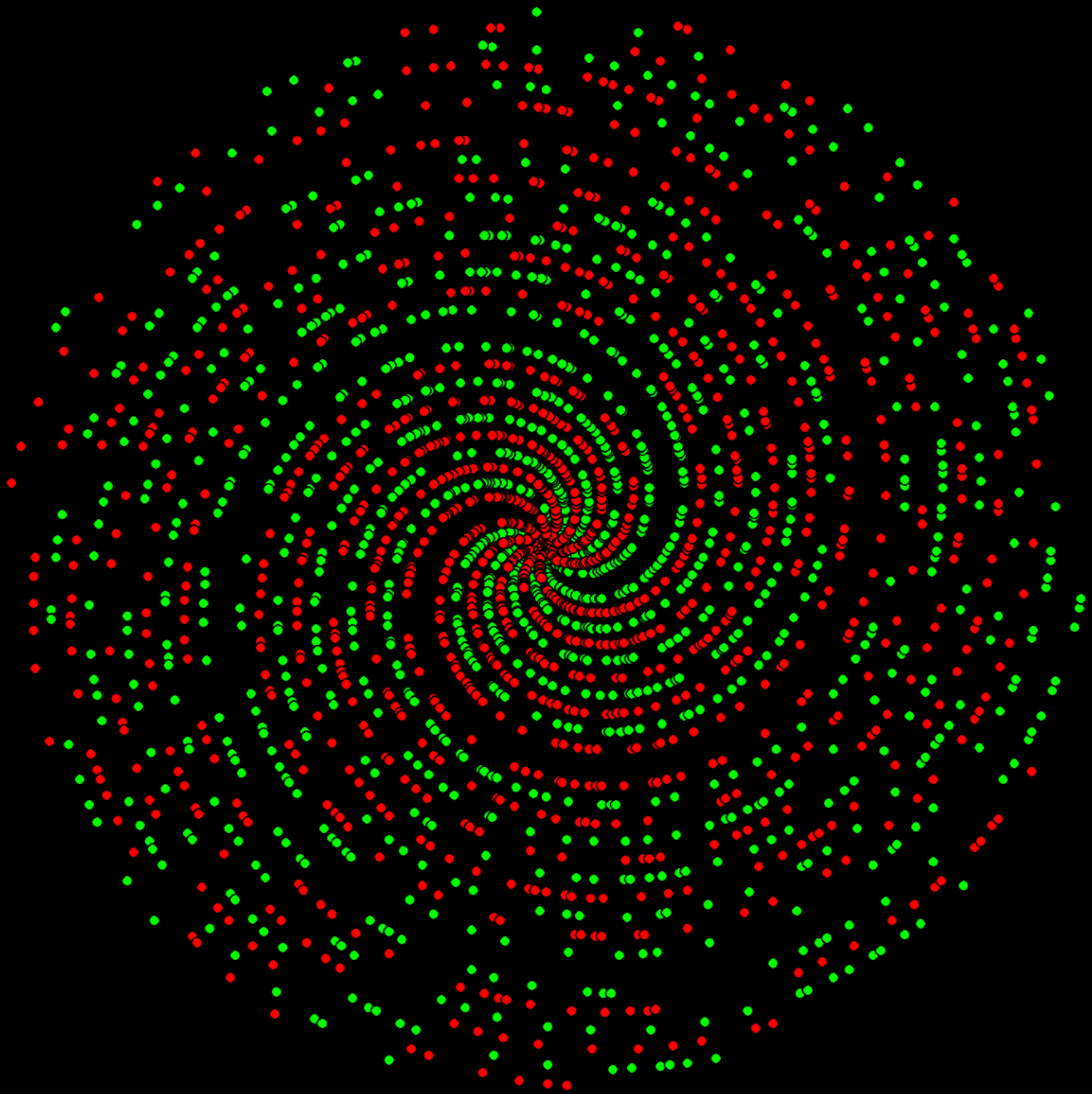
Primes generated
by $n^2 + n + 17$
(11, 13, 17, 23 ...)

Primes generated
by $n^2 + n + 17$
(11, 13, 17, 41 ...)



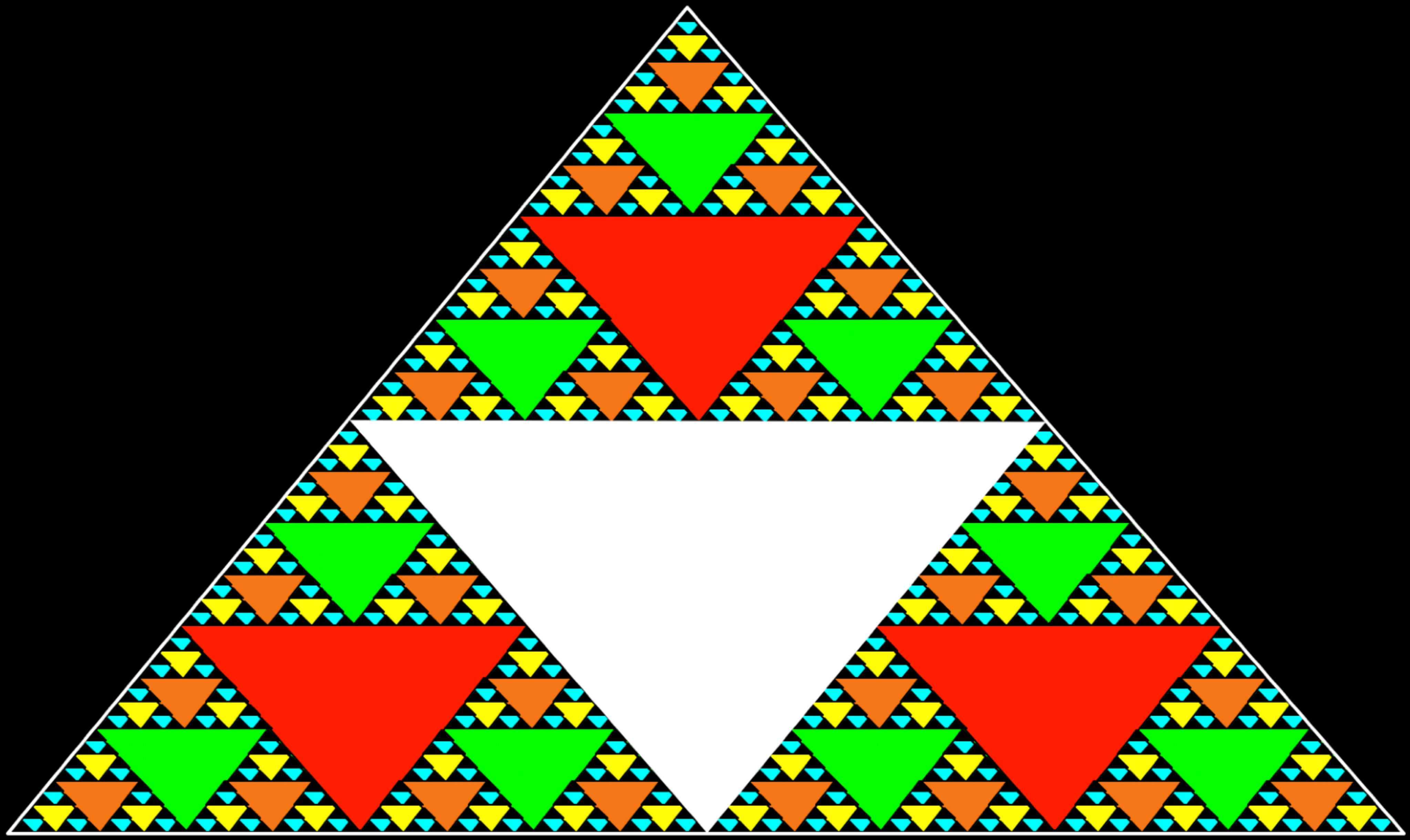
Primes plotted along Archimedian Spiral 2

Primes generated
by 4n +1
(5, 13, 17, 29 ...)

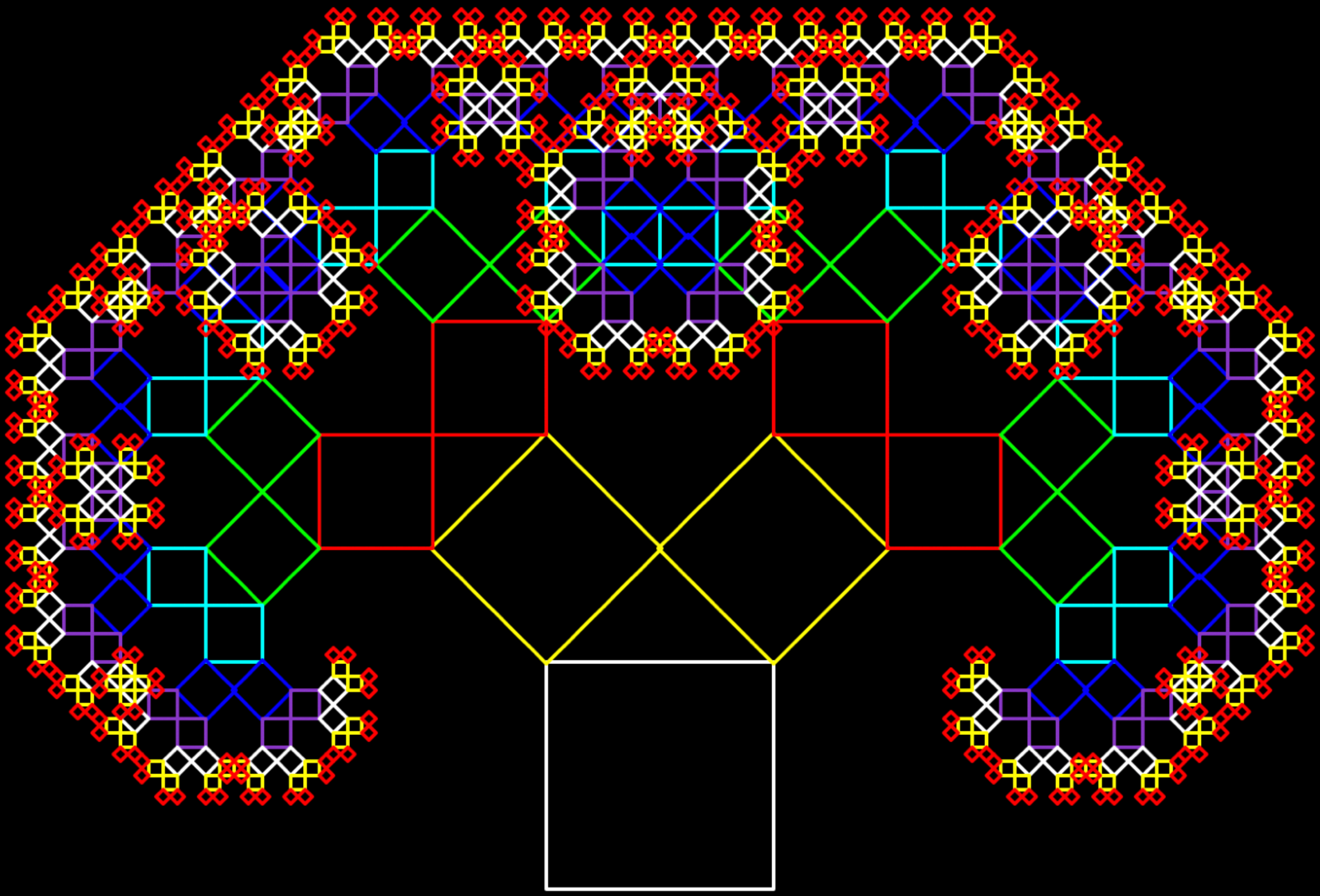


Primes generated
by 4n+3
(7, 11, 19, 23 ...)

Triangles and Trees



Sierpinski's Triangle - Progressively smaller triangles are created by connecting the midpoints of the existing triangles. This process continues indefinitely, but in this case is illustrated to a depth of only 6.



Pythagorean Tree - Progressively smaller squares are attached at 45° angles to the corners of the existing squares. This process should be continued indefinitely, but in this case is illustrated only to a depth of 9. The next page shows how numbering the squares in binary form can be used to create this fractal.

Underlying Mathematics

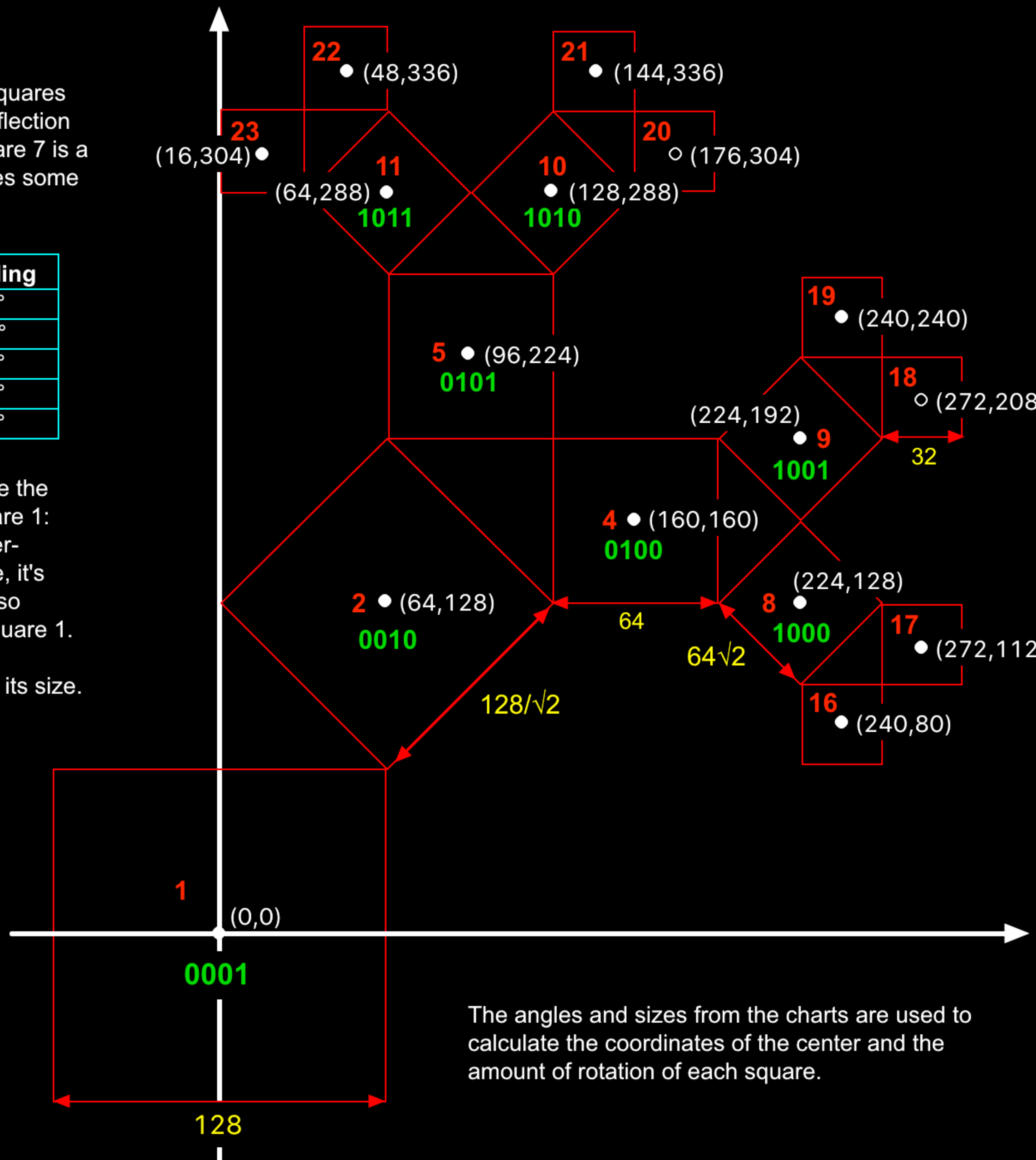
The squares are numbered as shown. The omitted squares are their reflections over the y-axis. (Square 3 is a reflection of square 2, square 6 is a reflection of square 4, square 7 is a reflection of square 5, etc.) The chart below expresses some of those numbers in binary form.

	Binary	Heading		Binary	Heading
1	0001	0°	6	0110	0°
2	0010	45°	7	0111	- 90°
3	0011	- 45°	8	1000	135°
4	0100	90°	9	1001	45°
5	0101	0°	10	1010	45°

The binary digits to the right of the left-most 1 indicate the rotations involved as you go to that square from square 1: 0 means rotate clockwise 45°; 1 means rotate counter-clockwise 45°. To get to square 9 (1001), for example, it's clockwise 45°, clockwise 45°, counterclockwise 45°, so square 9 ends up rotated 45° clockwise relative to square 1.

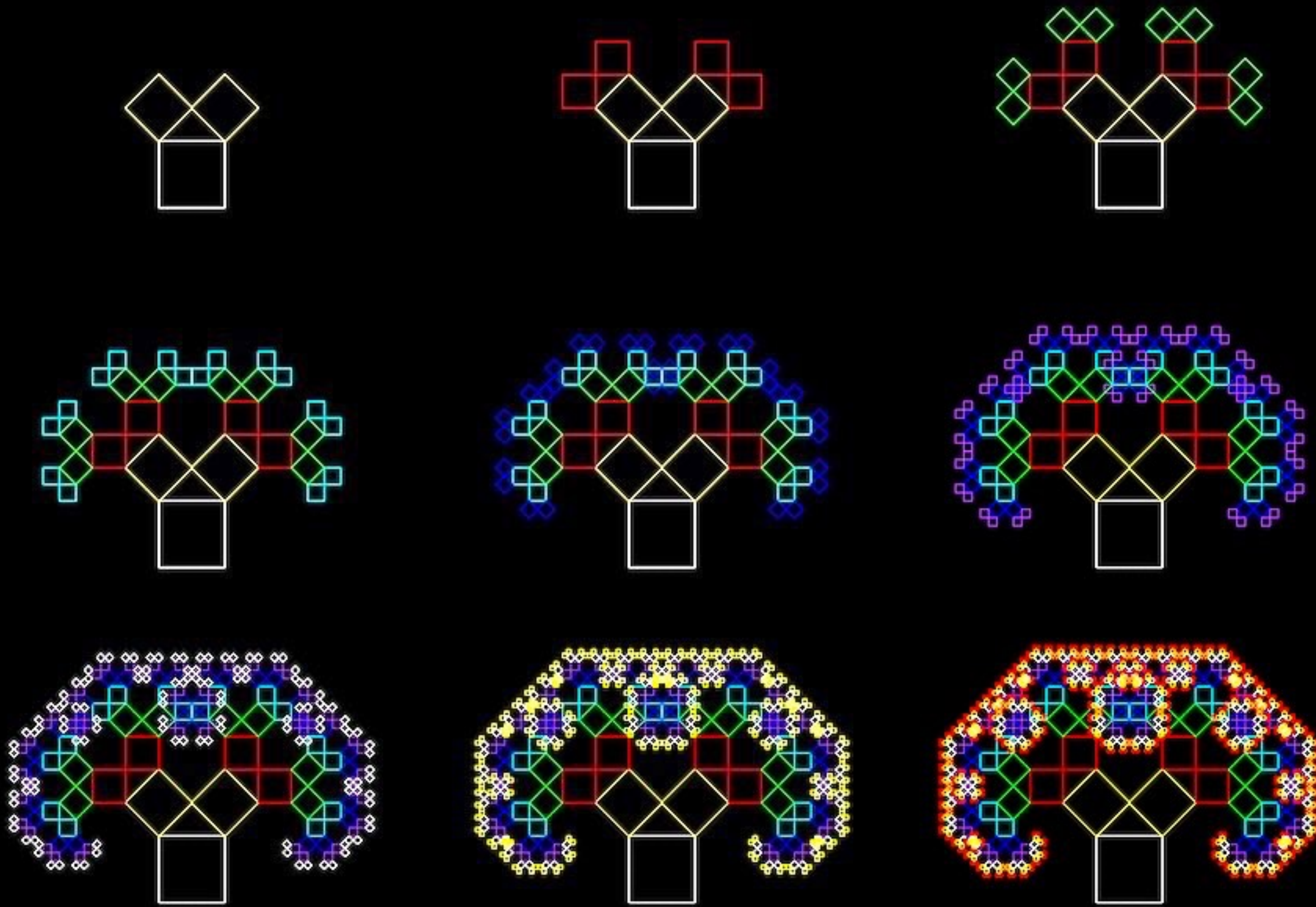
The digits of a square's binary number also indicates its size. Look for the left-most digit which is a 1.

	Binary	Digit	Size
1	0001	1's	128
2	0010	2's	$128/\sqrt{2}$
3	0011	2's	$128/\sqrt{2}$
4	0100	4's	64
5	0101	4's	64
6	0110	4's	64
7	0111	4's	64
8	1000	8's	$64/\sqrt{2}$
9	1001	8's	$64/\sqrt{2}$
10	1010	8's	$64/\sqrt{2}$

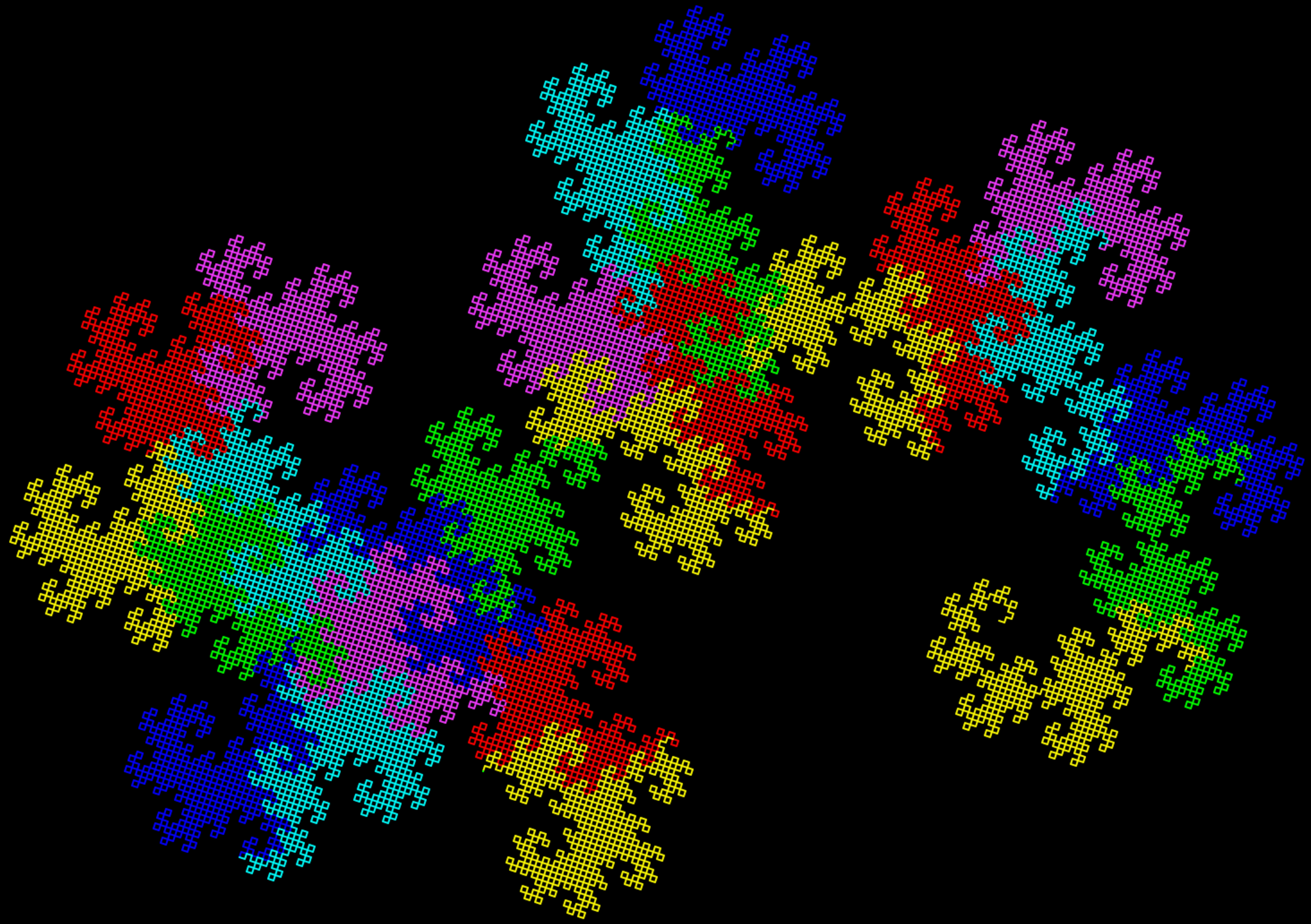


The angles and sizes from the charts are used to calculate the coordinates of the center and the amount of rotation of each square.

Pythagorean Tree - Growing from a depth of 1 to a depth of 9.

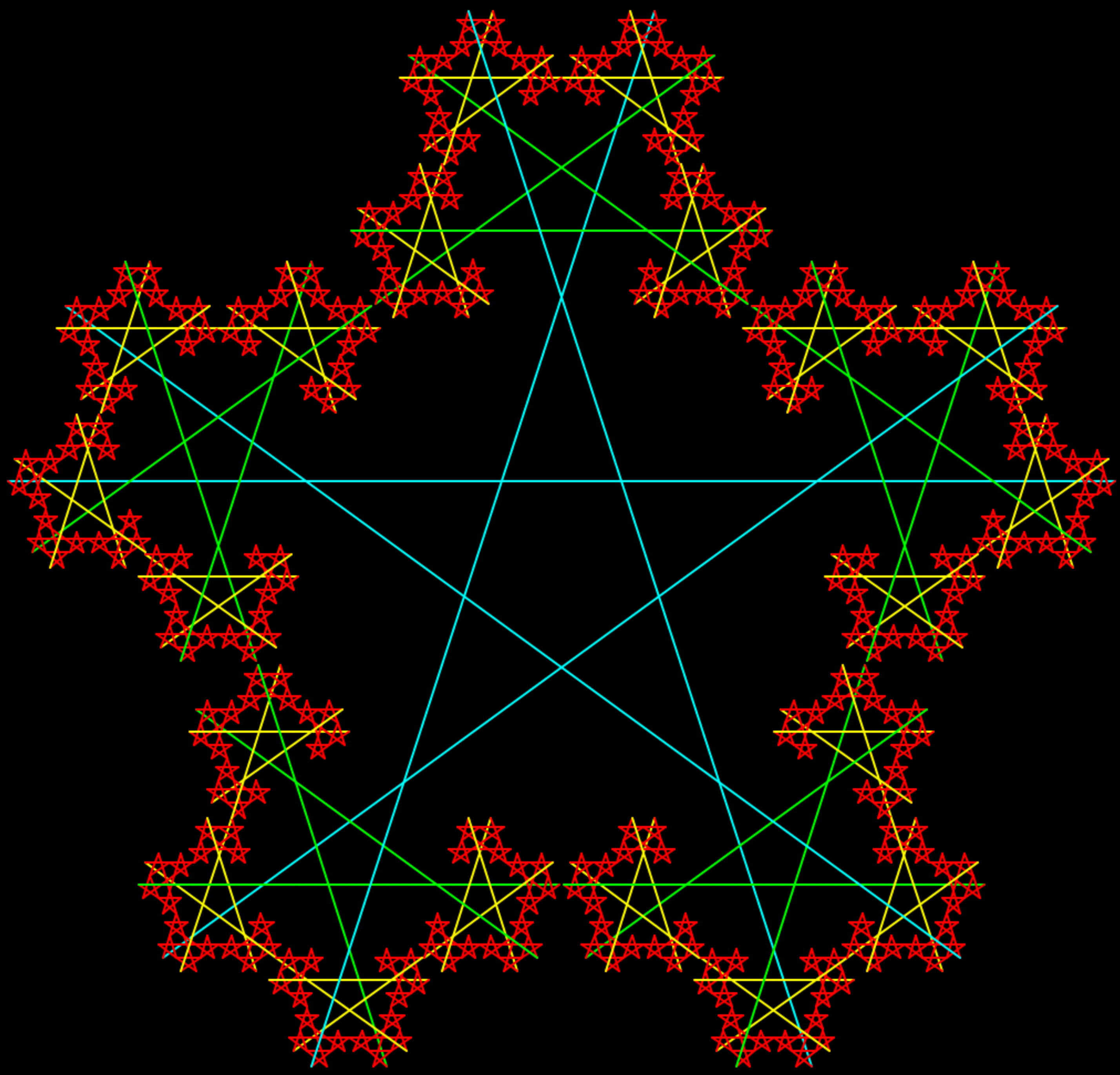


Dragons and Stars



Dragon Curve - This meandering line is formed by repeatedly folding a strip of paper in the same direction and then unfolding it to form 90° angles at the folds. In this case 25,000 digital folds were made.

Star Fractal - another meandering line



Newton's Method

Newton's Method is a recursive process for finding the roots (zeros) of a function with as much precision as desired. The function y = f (x), graphed here, crosses the x-axis at some unknown point. The goal is to find the x-coordinate of that point.

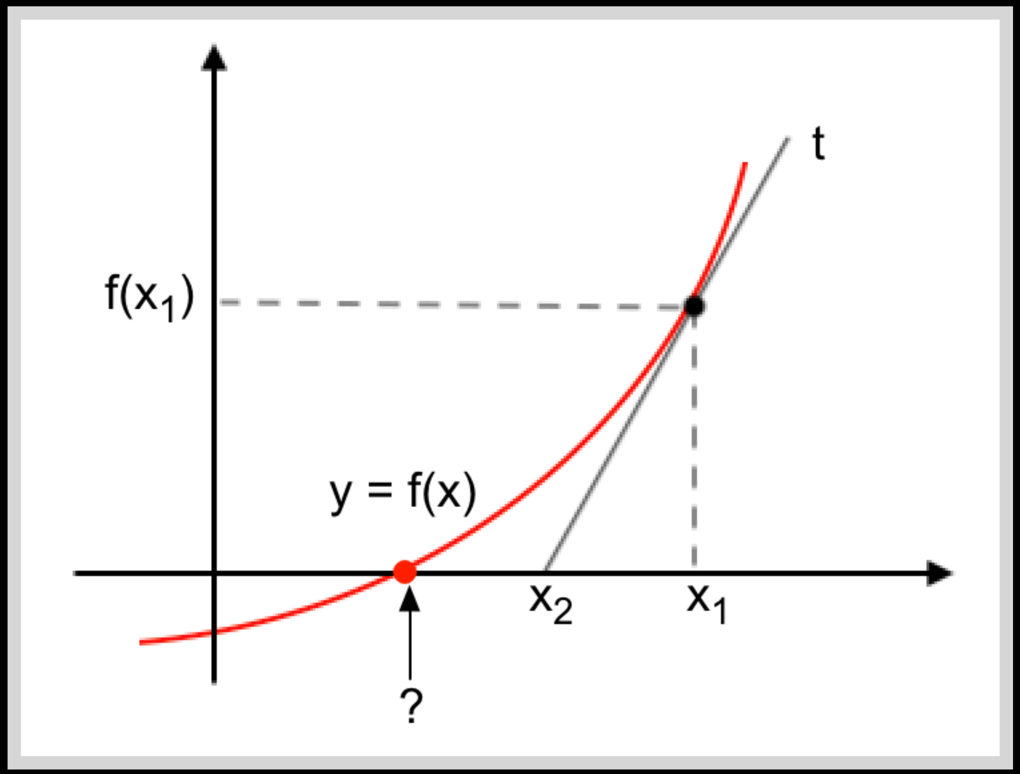
You start by guessing an answer. Suppose your guess is x_1 (a pretty bad guess). You then find the point (x_1,f (x_1)) on the graph of the function, draw the tanget to the curve at that point, and see where it crosses the x-axis. Suppose it crosses at x_2. That will be your next guess. Subsequent guesses will get you even closer to the actual zero. The general recursive formula is:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

where f ′ is the derivative of f.

Whether the formula actually converges to a zero (and which zero if there is more than one) depends on the initial guess.

Finding the real zeros of f (x)



Although we can't illustrate the process with a 2-dimensional drawing, the same recursive formula can be used to find the complex zeros of polynomial functions. Which initial guesses lead to which zeros, however, can be surprising and beautiful.

Finding the Cube Roots of 1

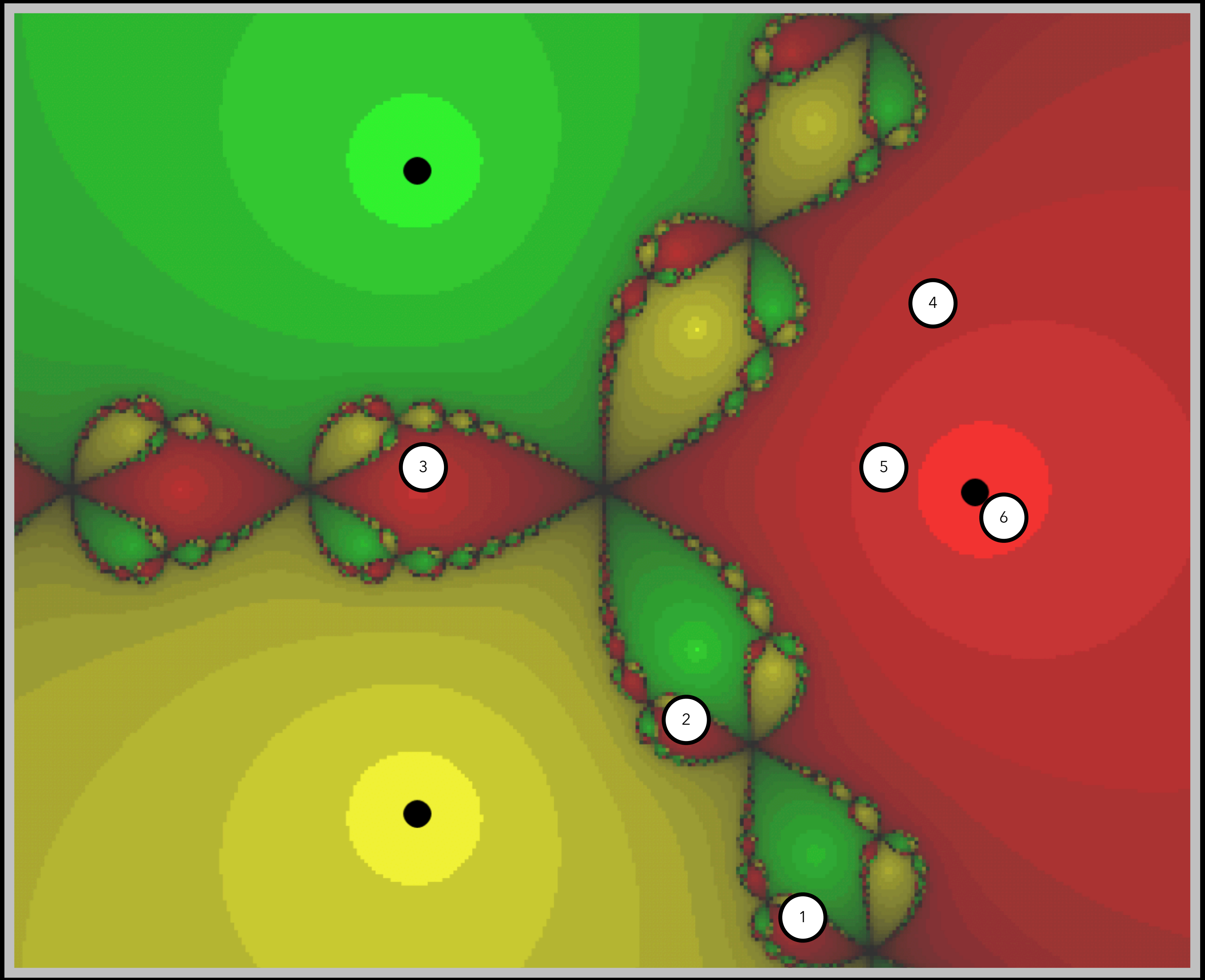
The complex plane shown here is centered at 0 + 0i. The three black spots mark the location of the three zeros of $f(x) = x^3 - 1$. They are at approximately 1 + 0i, - .5 + .866i, and -.5 - .866i.

If the initial guess is in any red area, the recursive formula will converge to 1 + 0i.

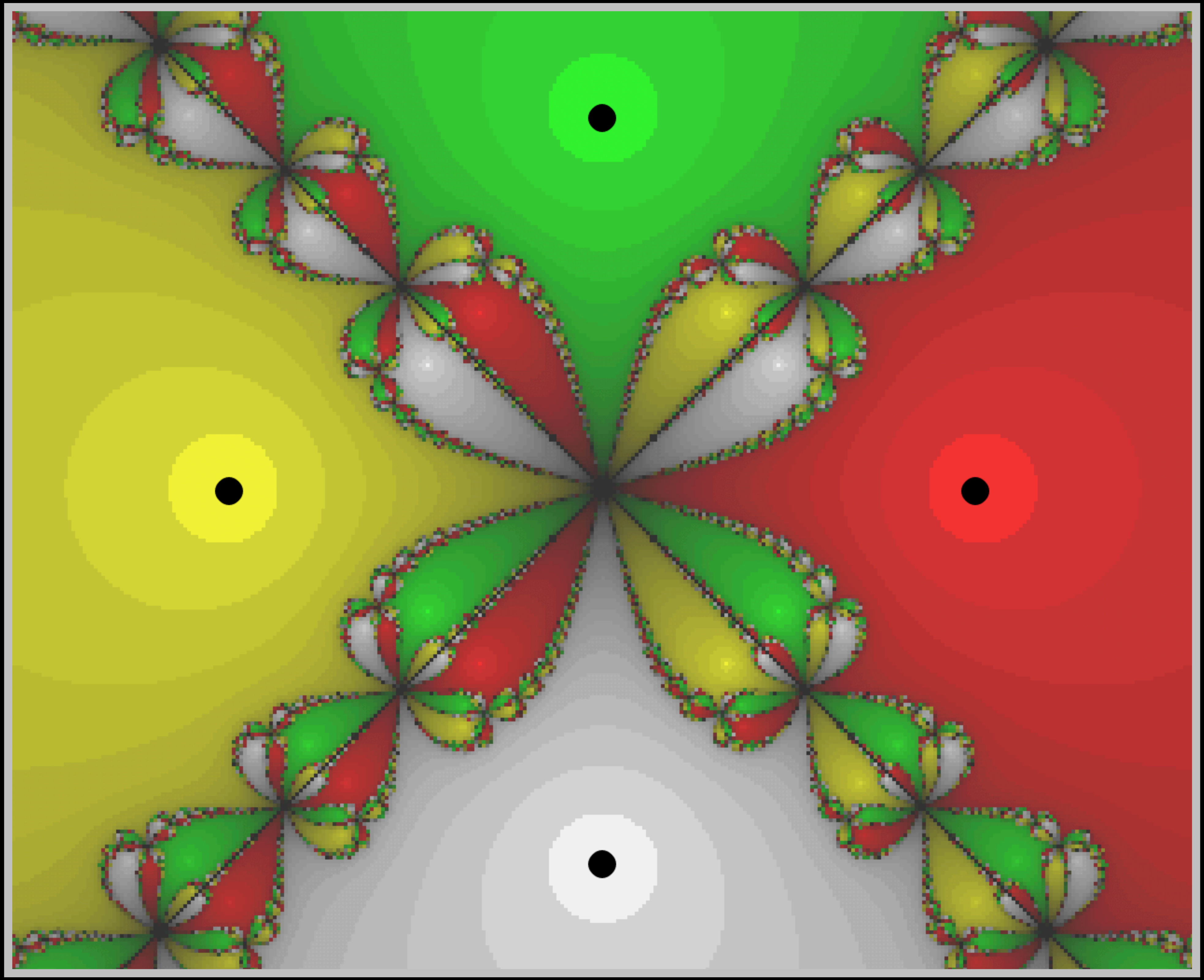
If the initial guess is in any green area, the recursive formula will converge to -.5 + .866i.

If the initial guess is in any yellow area, the recursive formula will converge to -.5 - .866i.

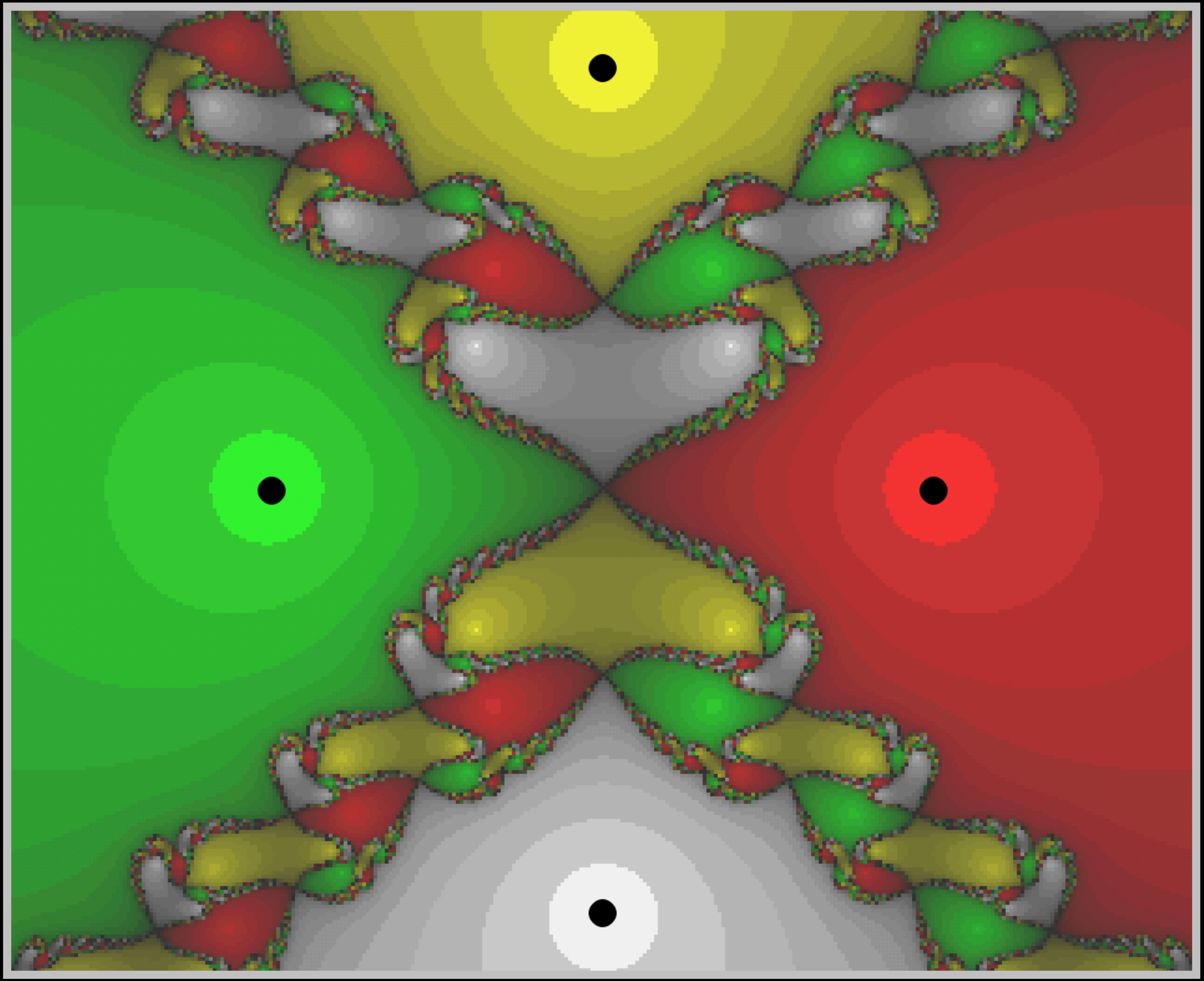
The numbered dots show the iterations of the recursive function when starting with an initial guess of approximately .525 - 1.65i.



Finding the 4th Roots of 1



Finding the Roots of $f(x) = x^4 + .5x - 1$



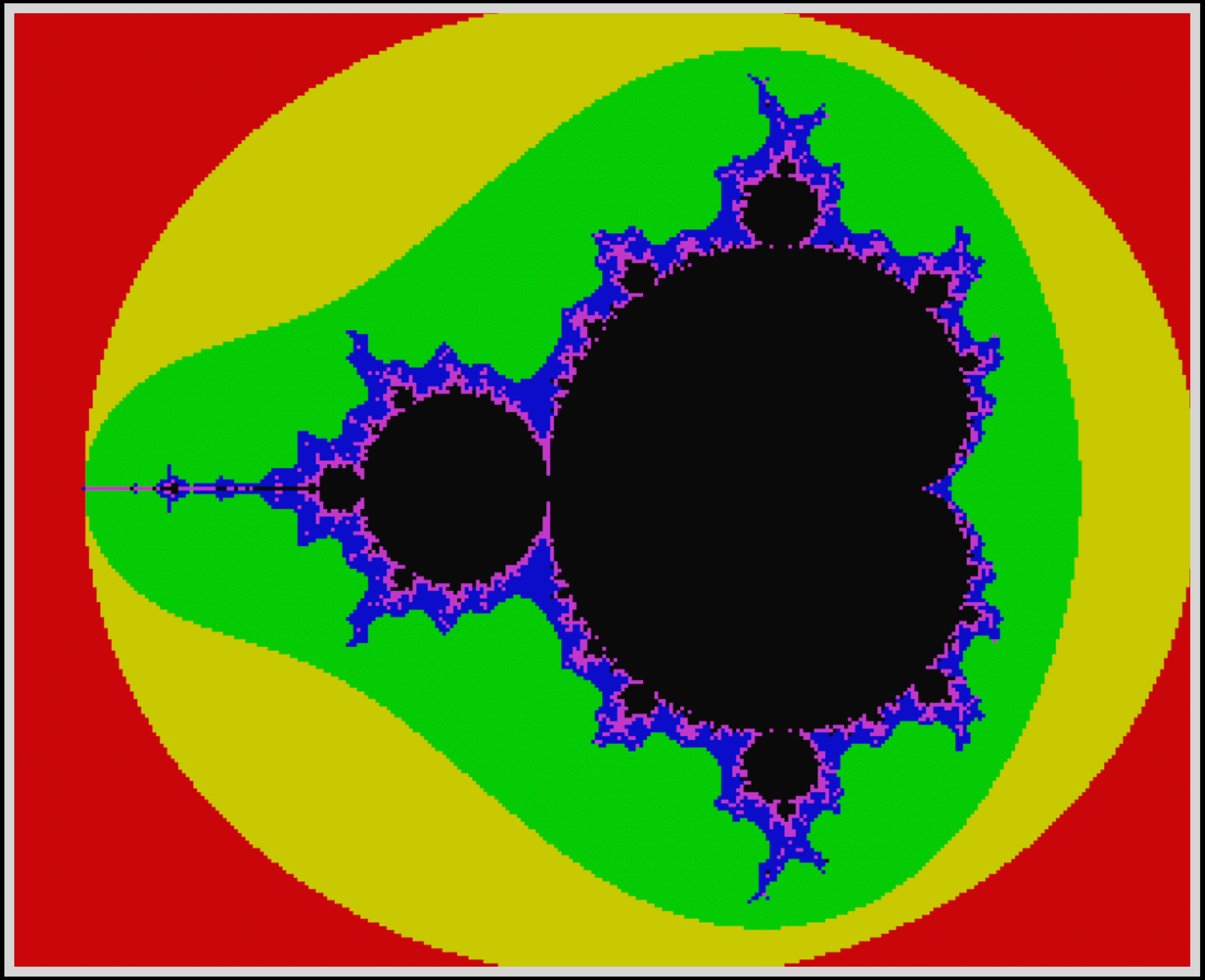
Mandelbrot Set

The complex plane shown here is centered at .6 + 0i. The real axis extends from -2.2 to 1, and the imaginary axis extends form -1.3i to 1.3i.

Let x_0 = 0+0i and let c be any complex number in the plane. Use the recursive formula $z_{n+1} = (z_n)^2 + c$ to see what happens after many iterations.

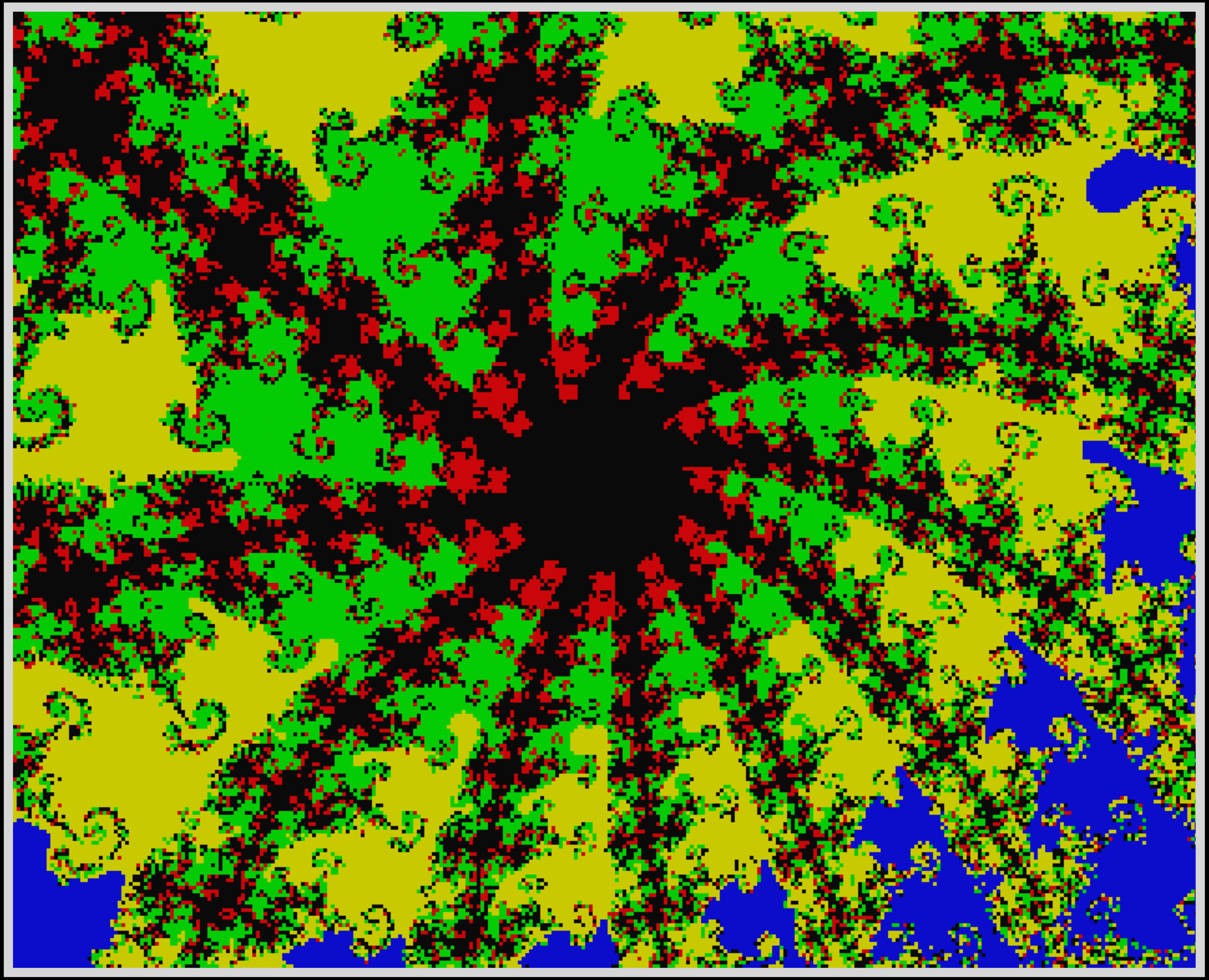
If the absolute value of z_n gets larger and larger, then the point at c is given a color indicating just how quickly that happened.

If, on the other hand, the absolute value remains bounded after 100 iterations, the point is colored black. This is the Mandelbrot Set.



Mandelbrot Close-up

(Centered at -.78386+.137411i, Magnified 65536 times, 400 iterations)



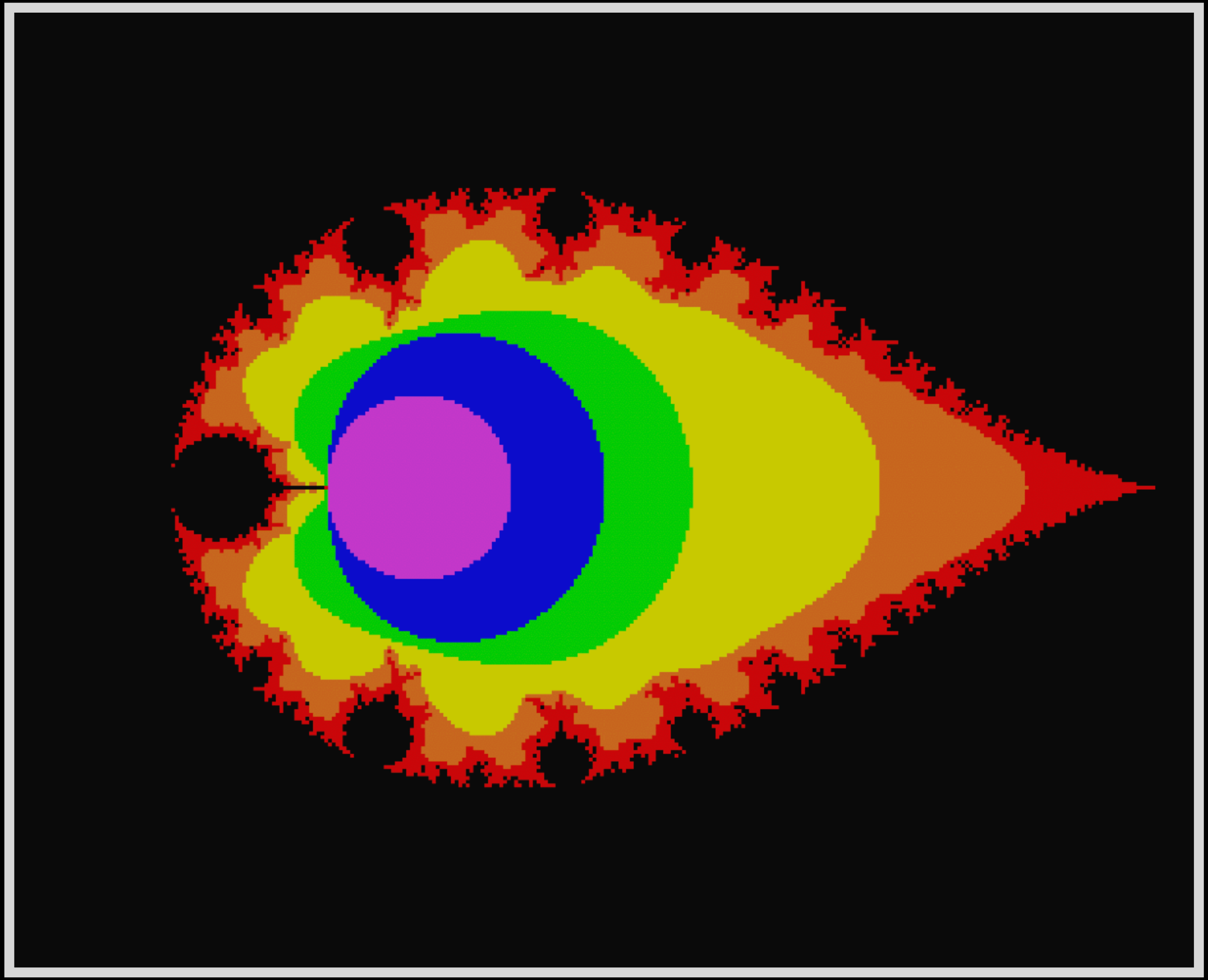
Inverse Mandelbrot Set

The complex plane shown here is centered at 1.0 + 0i. The real axis extends from -2.2 to 4.2, and the imaginary axis extends form -2.6i to 2.6i.

Let x_0 = 0+0i and let c be any complex number in the plane. Use the recursive formula $z_{n+1} = (z_n)^2 + 1/c$ to see what happens after many iterations.

If the absolute value of z_n gets larger and larger, then the point at c is given a color indicating just how quickly that happened.

If, on the other hand, the absolute value remains bounded after 100 iterations, the point is colored black. The result is called the inverse Mandelbrot set.



Julia Sets

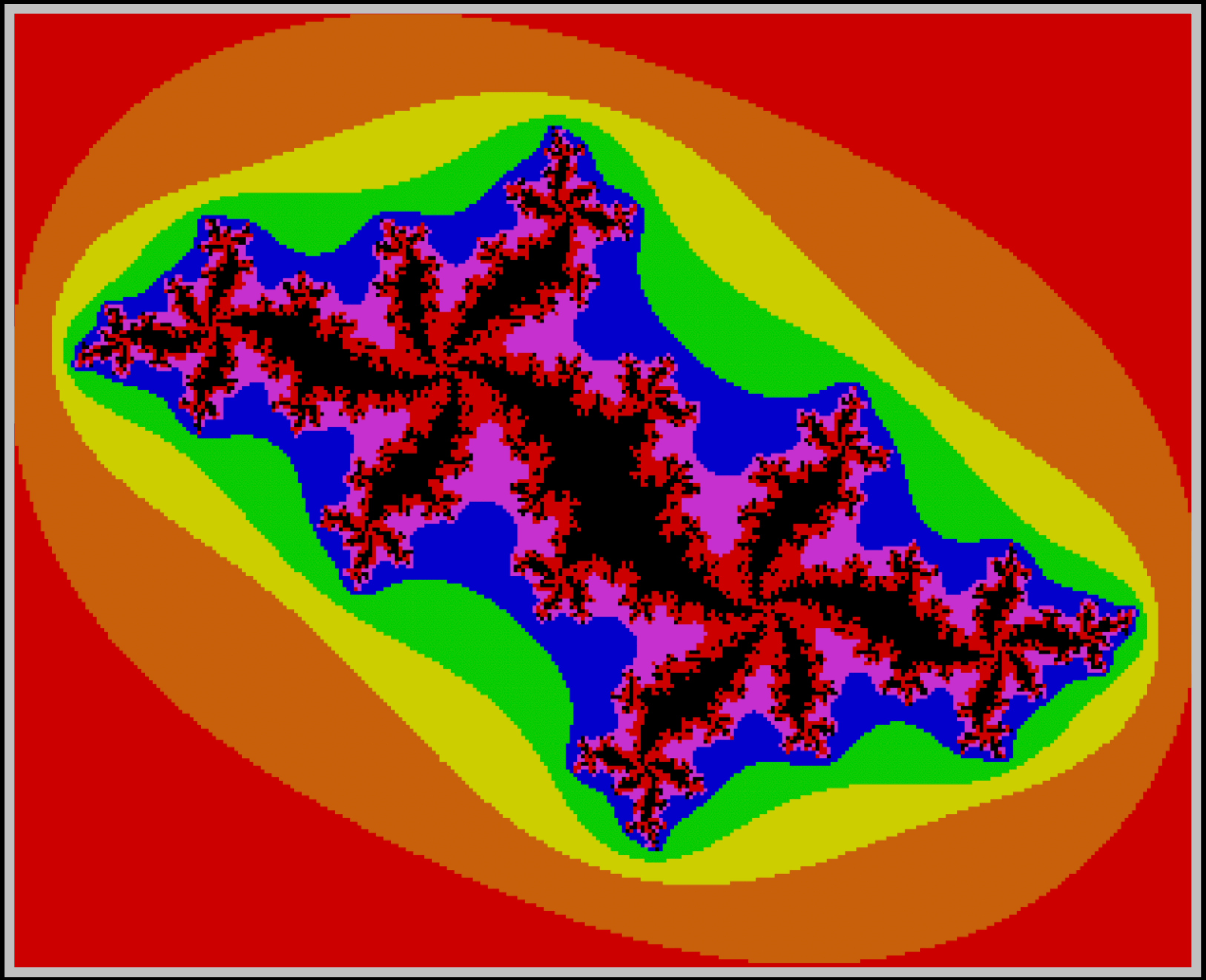
The complex plane shown here is centered at 0 + 0i. The real axis extends from -1.6 to 1.6, and the imaginary axis extends form -1.3i to 1.3i.

Let z_0 be any complex number in the plane, and use the recursive formula $z_{n+1} = (z_n)^2 + c$, where c is a constant complex number, to see what happens after many iterations.

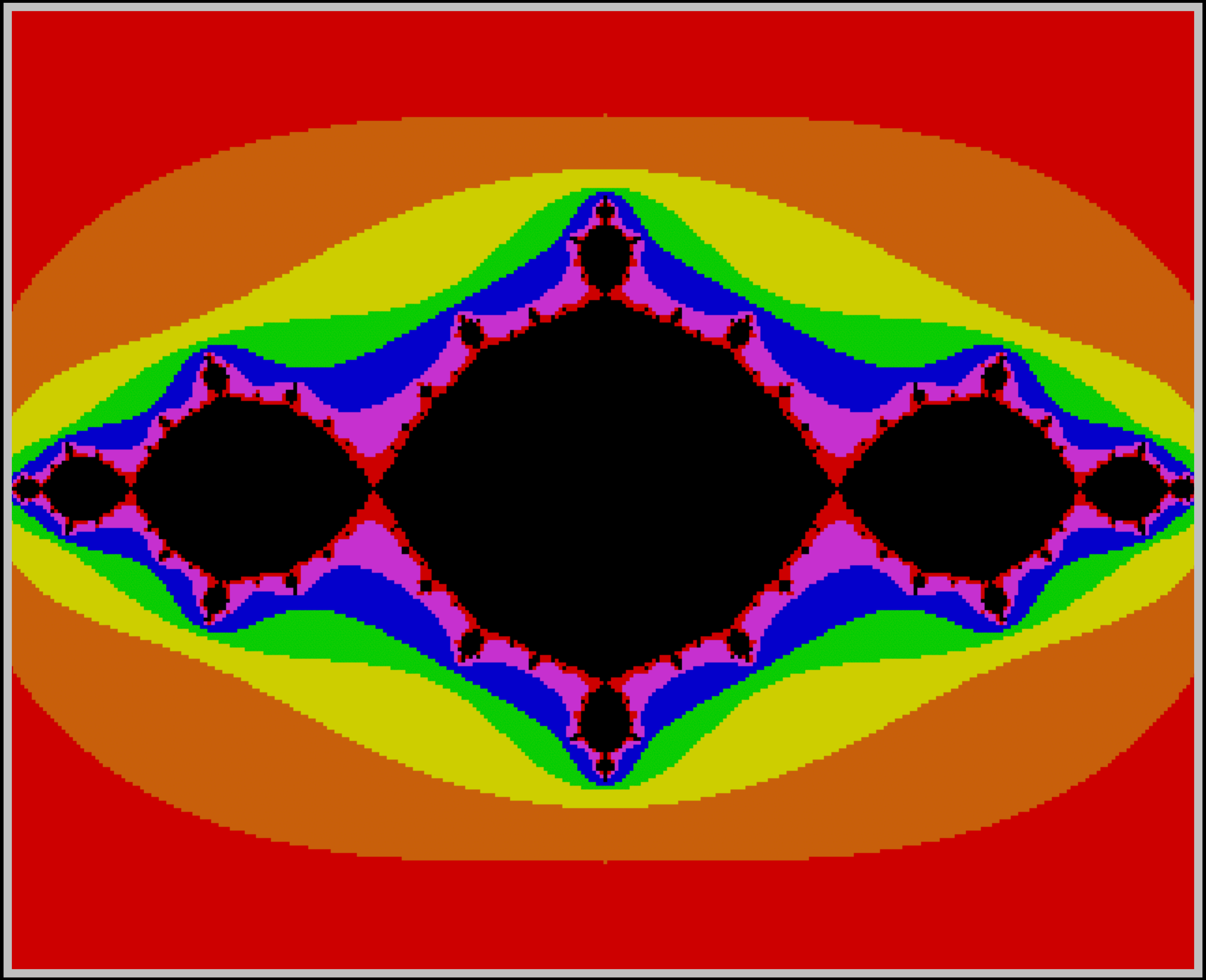
If the absolute value of z_n gets larger and larger, then the point at z_0 is given a color indicating just how quickly that happened.

If, on the other hand, the absolute value remains bounded after 100 iterations, the point is colored black. The black points form a Julia Set.

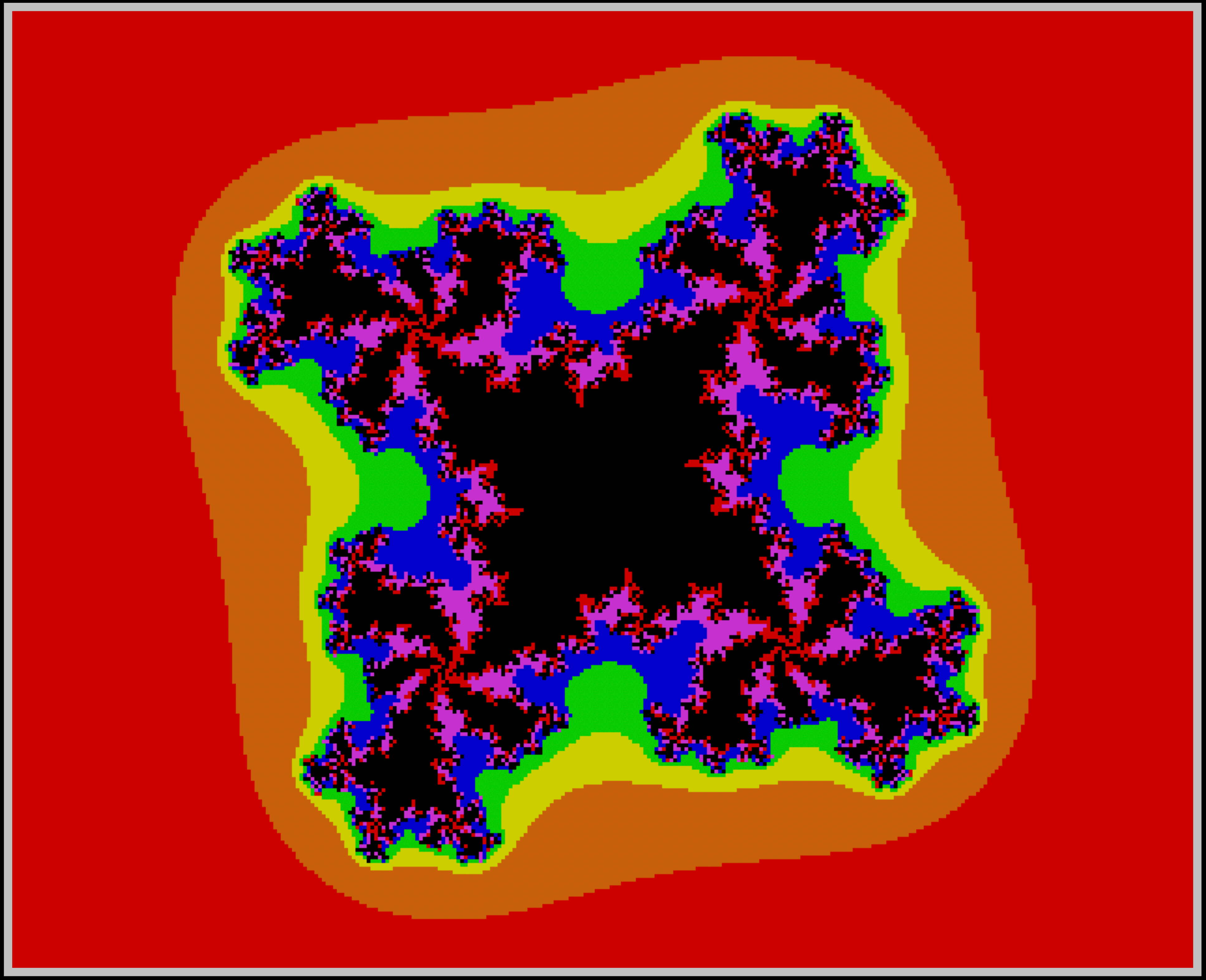
Julia Set when c = -.5 +.6i



Julia Set when $z_{n+1} = (z_n)^2 - 1$

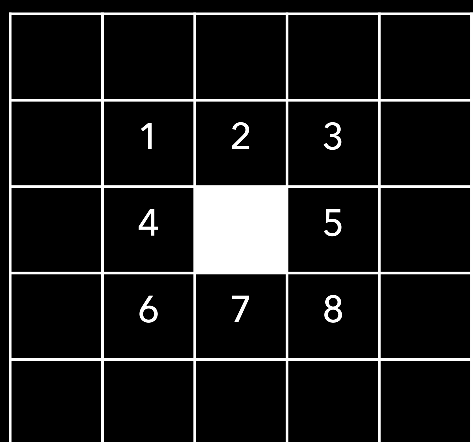


Julia Set when $z_{n+1} = (z_n)^4 + .6 + .535i$

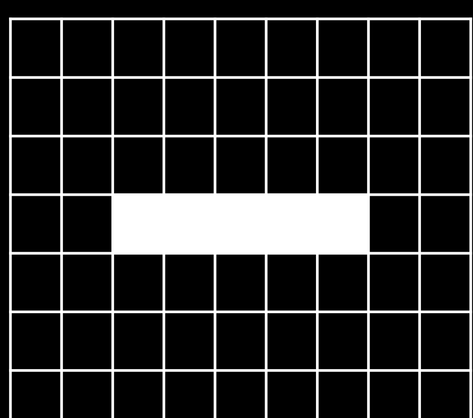


Automata - One and Eight Rule

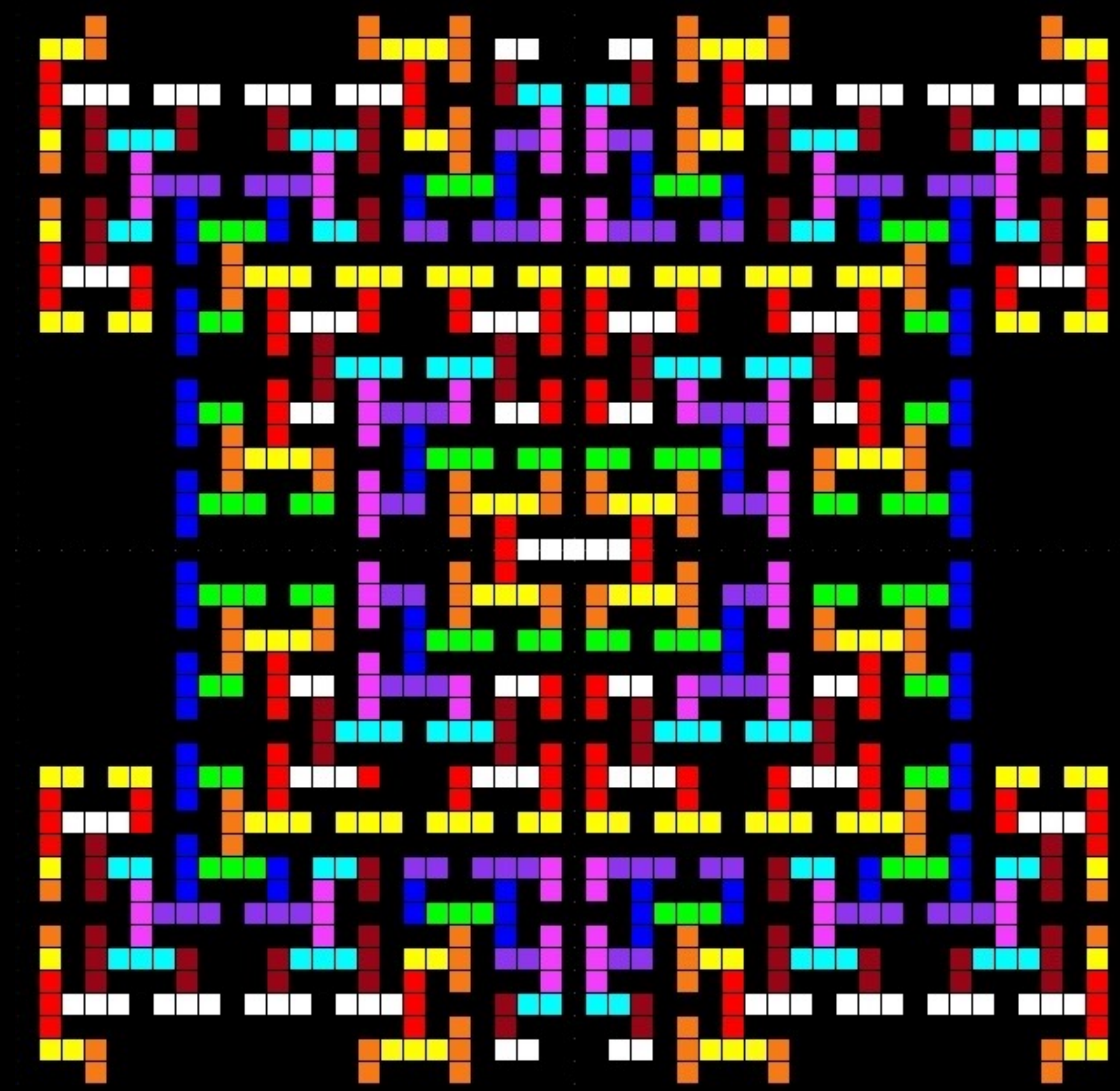
Each cell on a large grid has 8 neighboring cells. Each of those cells can be either “alive” or “dead.” An initial pattern of living cells changes recursively according to this rule: A cell comes alive (becomes colored) if exactly one of its neighbors is currently alive; otherwise it stays the same as it was. Here’s what happens if the initial pattern is 5 living cells in a row. This is what it looks like after 30 iterations.



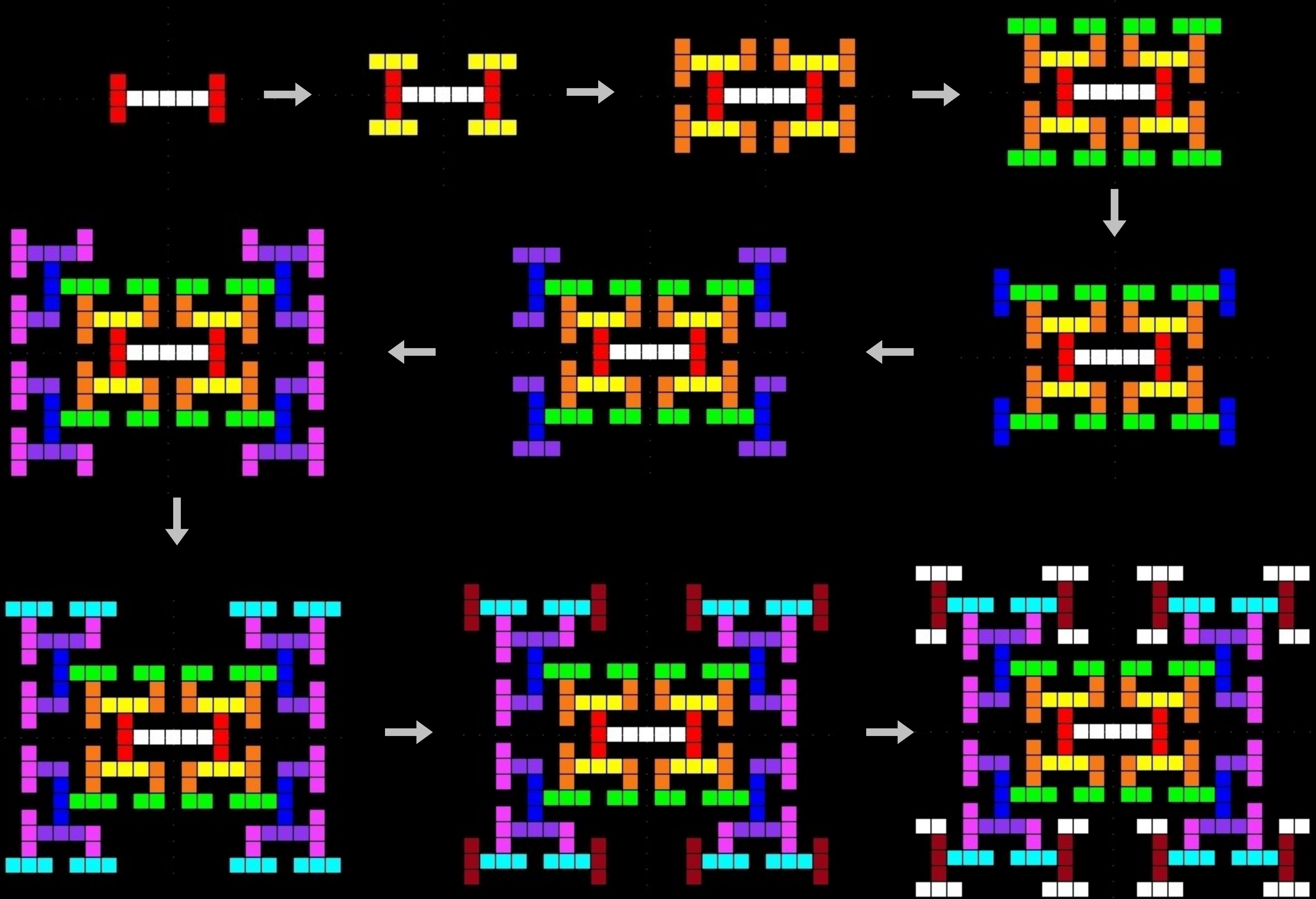
8 Neighbors



Initial Pattern

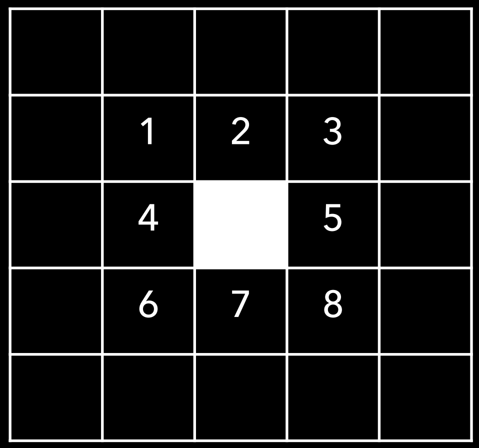


The first 10 iterations

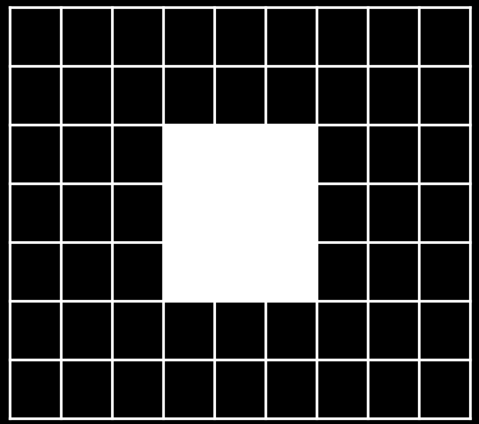


Automata - another One and Eight Rule example

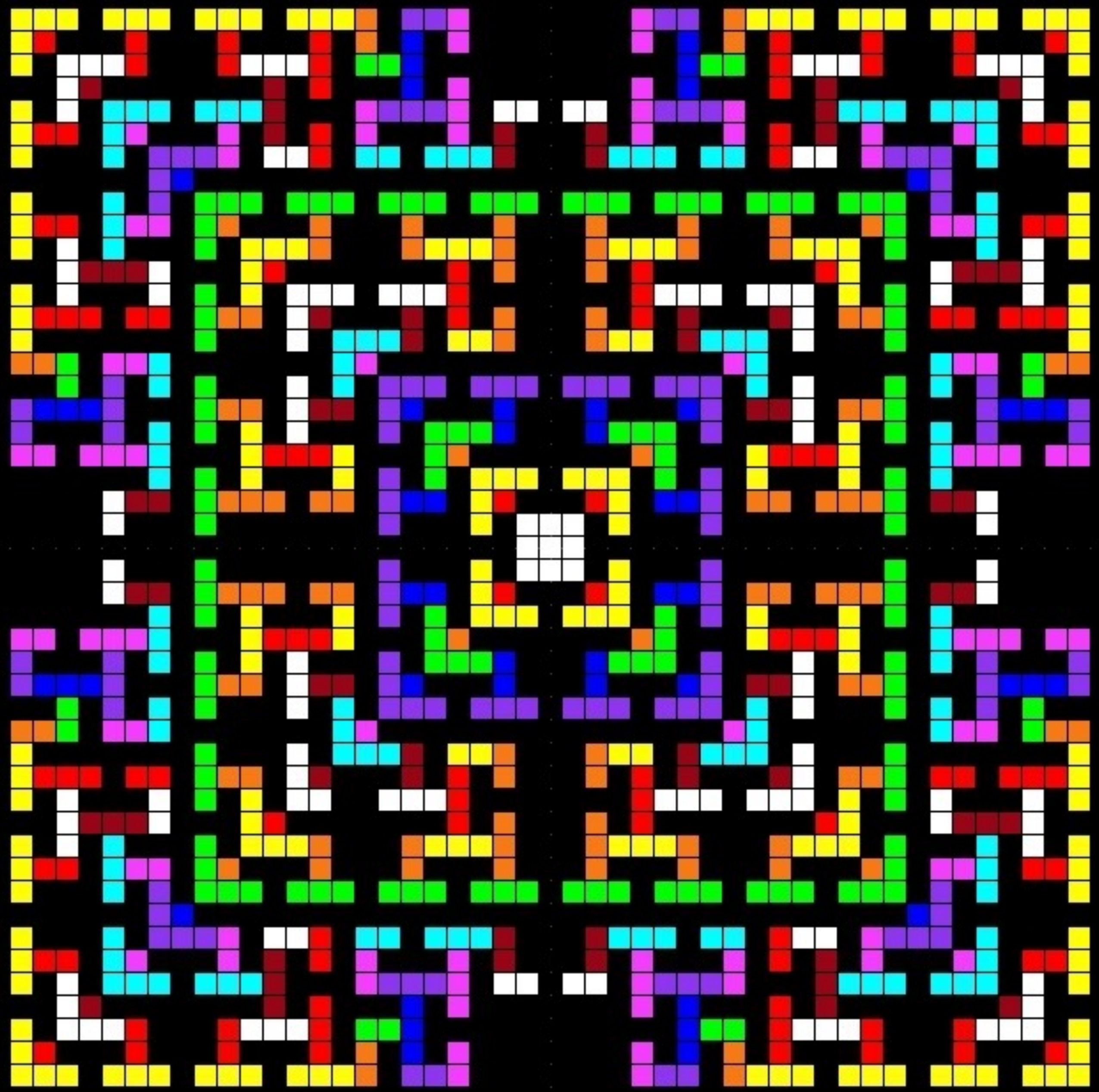
Another example following the same rule, this time starting with a 3x3 square pattern of living cells. Note: The color of the new living cells changes each time the rule is applied. Previously existing cells retain their original color. Each "generation" shares a common color. This is what it looks like after 30 iterations.



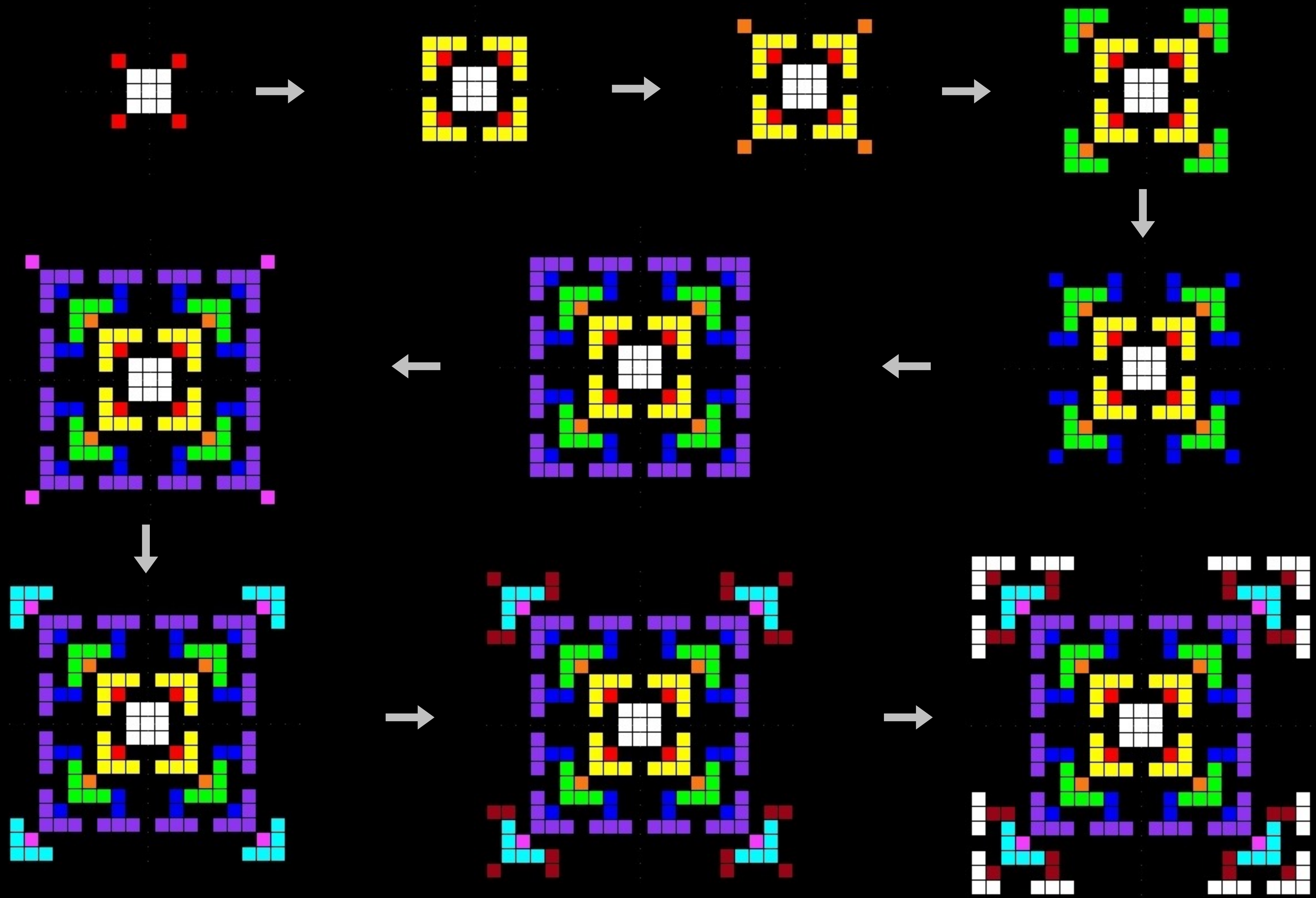
8 Neighbors



Initial Pattern

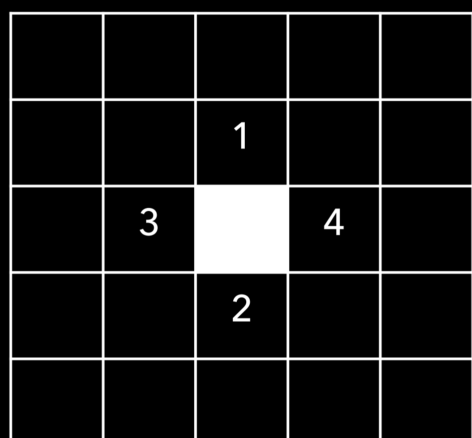


The first 10 iterations

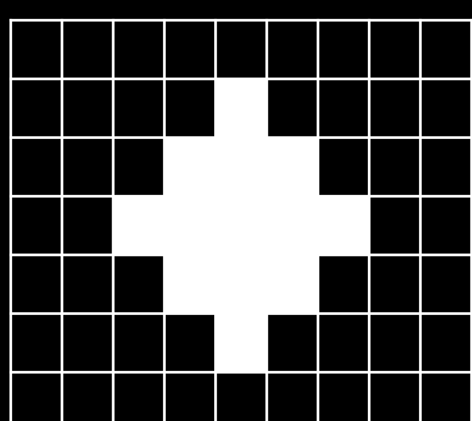


Automata - Parity Rule

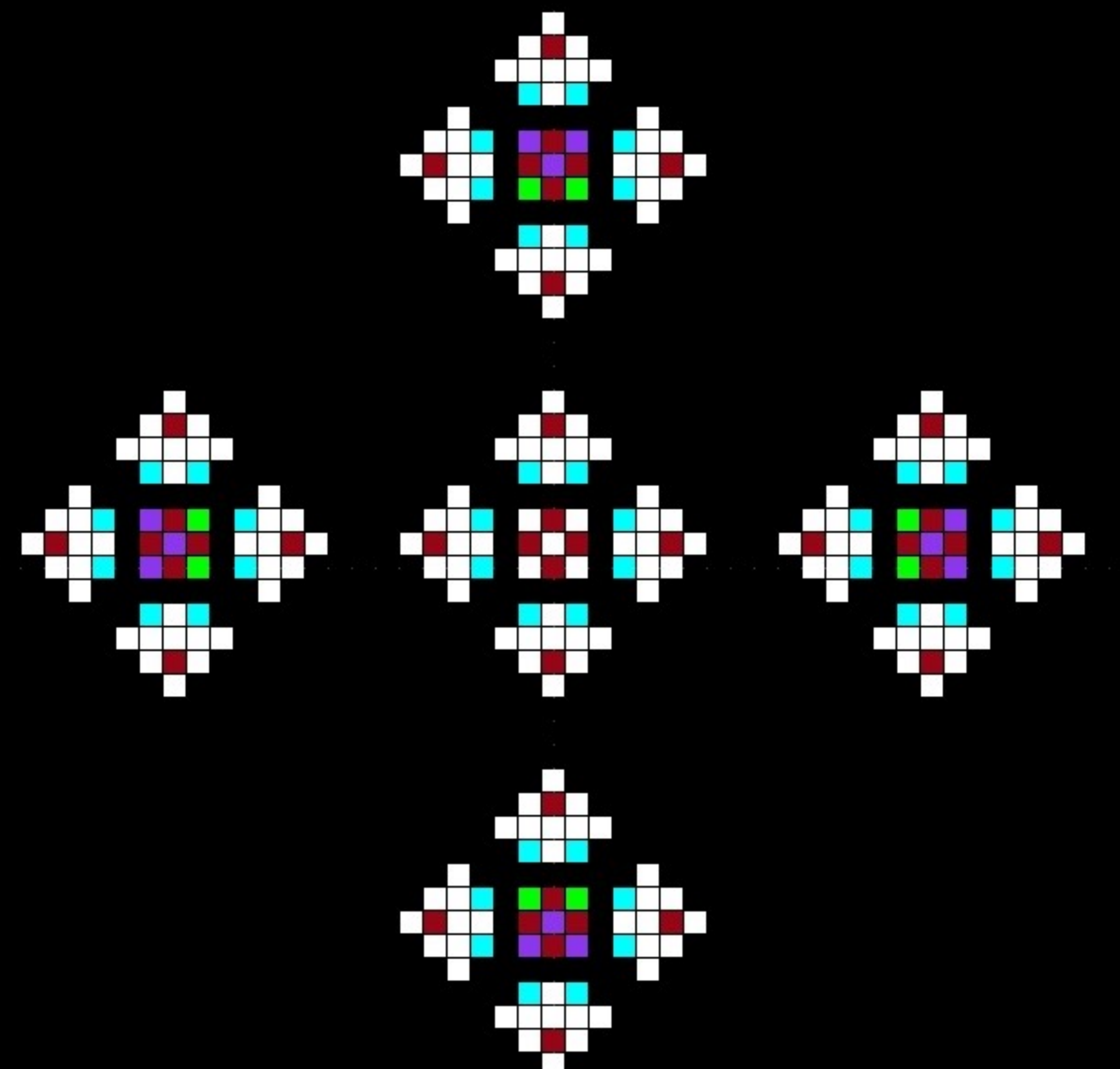
The fate of a cell depends only on its current state and the states of the four neighbors directly above, below, to the left, or to the right of it. Living cells are assigned a value of 1; non-living cells are assigned a value of 0. If their sum is an odd number, then the middle cell stays or becomes alive; otherwise, it dies or remains dead. This is what it looks like after 30 iterations.



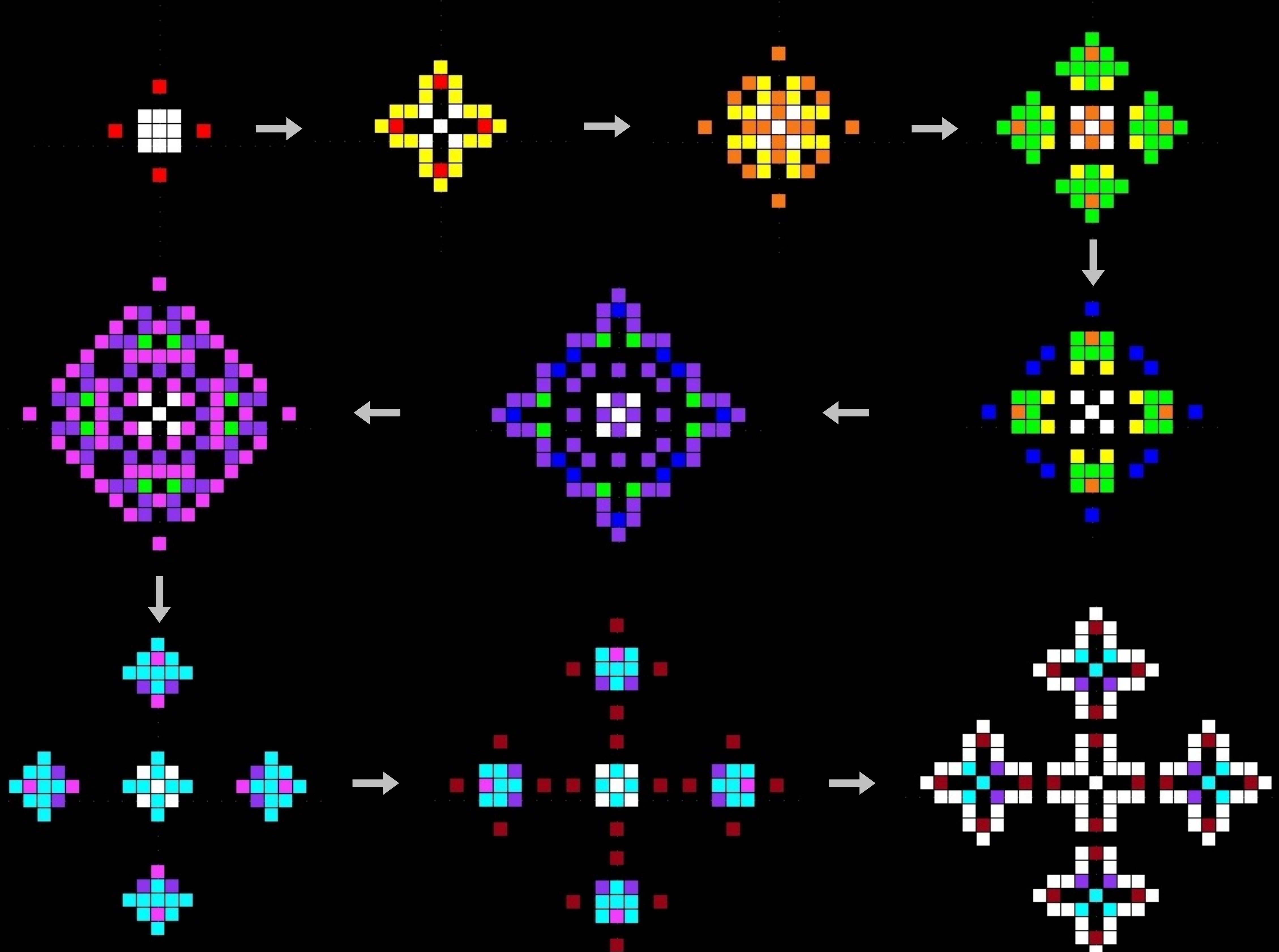
4 Neighbors



Initial Pattern

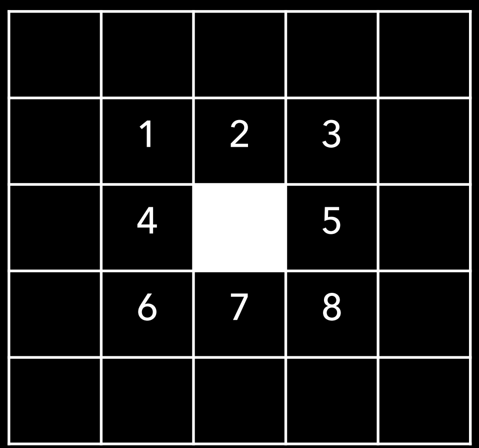


The first 10 iterations

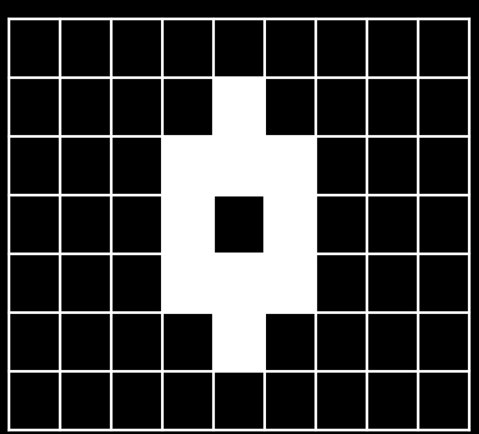


Automata - Life Rule

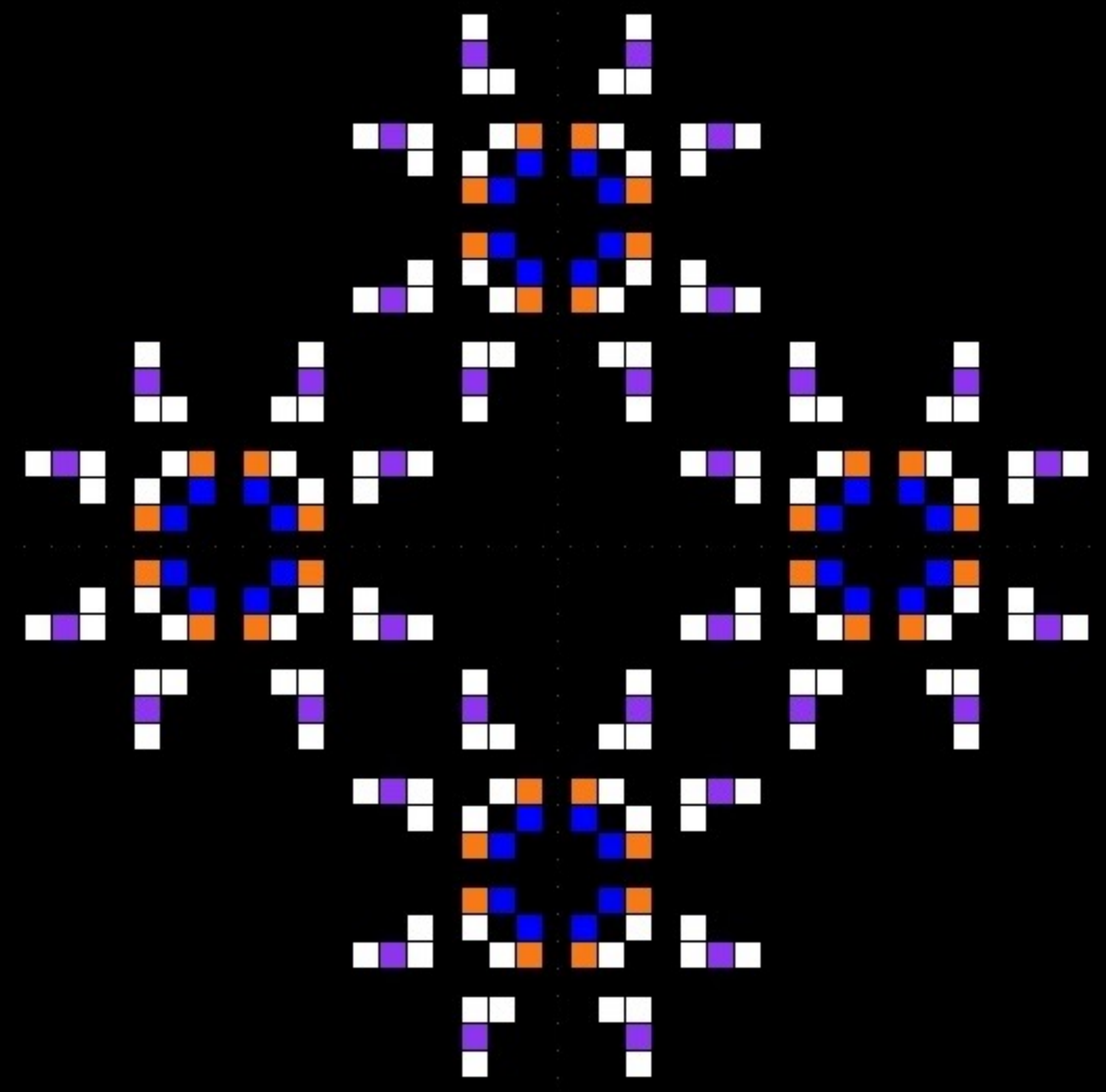
A living cell stays alive only when exactly two or three of its neighbors were alive in the previous generation. An non-living cell is “born” when exactly three of its neighbors were alive in the previous generation. Four of the shapes shown below make up the initial pattern. Four oscillating pulsars eventually appear and remain forever. This is what it looks like after 40 iterations.



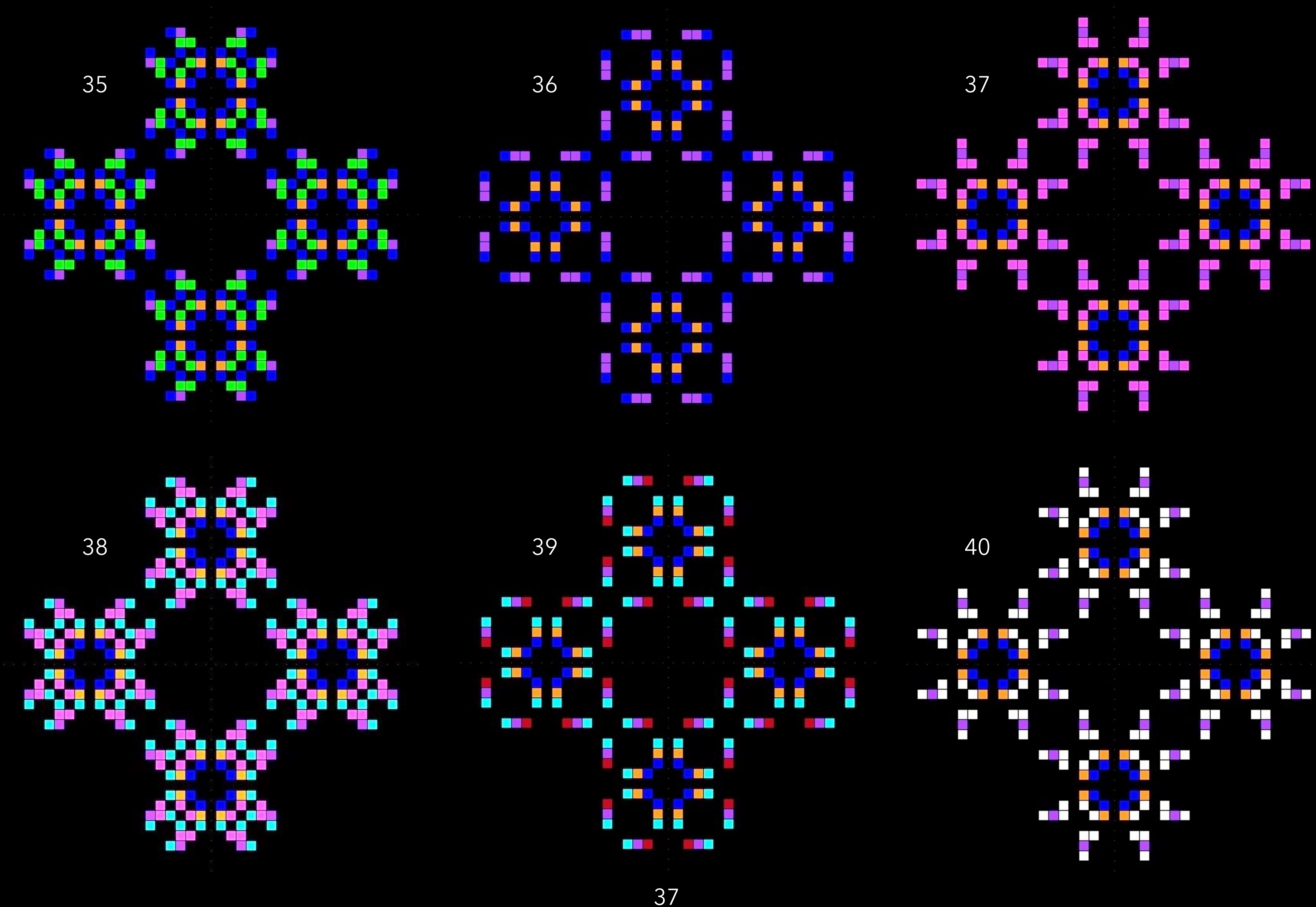
8 Neighbors



Part of Initial Pattern

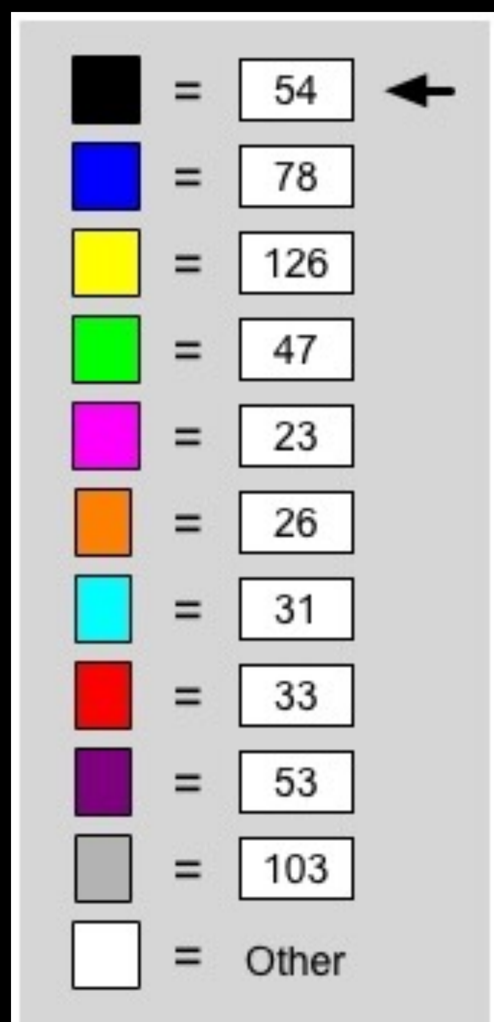


Iterations 35 to 40 - these three patterns repeat forever



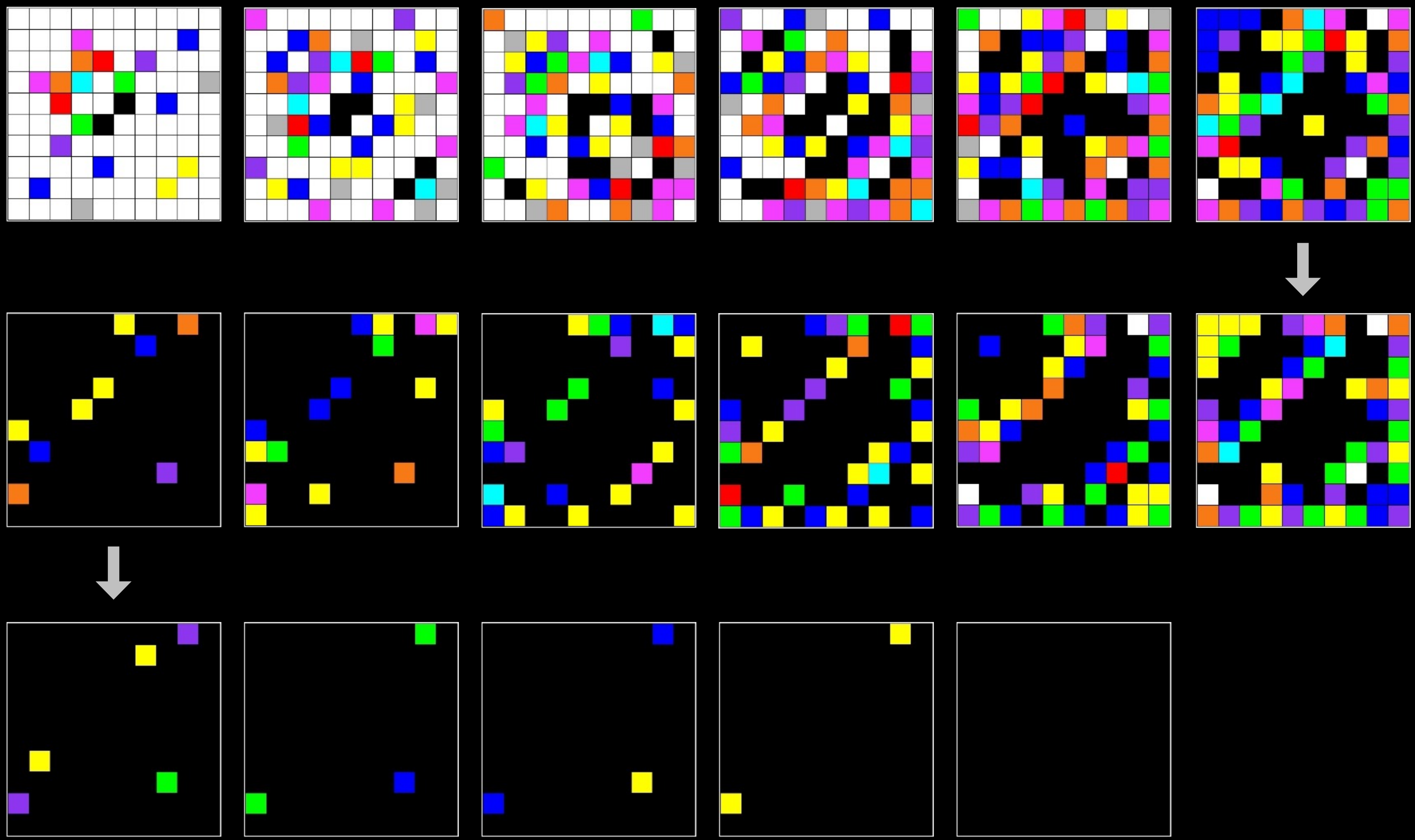
Black Holes - Squaring and Adding Digits

Start with the positive whole numbers between 0 and 99 arranged in a grid as shown below. For each number in the grid, add the squares of its two digits and then add 13. Replace the original number in the grid with the result. Repeat the process over and over again. If you get a 3-digit number, simply add the squares of all three digits and add 13 to the result. What happens in the long run? The next page shows what happens through 17 iterations. Instead of actually displaying the numbers, color codes are used for the results that occur most often.



00	01	02	03	04	05	06	07	08	09
10	11	12	13	14	15	16	17	18	19
20	21	22	23	24	25	26	27	28	29
30	31	32	33	34	35	36	37	38	39
40	41	42	43	44	45	46	47	48	49
50	51	52	53	54	55	56	57	58	59
60	61	62	63	64	65	66	67	68	69
70	71	72	73	74	75	76	77	78	79
80	81	82	83	84	85	86	87	88	89
90	91	92	93	94	95	96	97	98	99

After 17 iterations every starting number gets stuck on black hole 54, here represented by a small black square. Nothing changes after you reach 54 because $5^2 + 4^2 + 13 = 25 + 16 + 13 = 54$.



Multiplying, Adding, and Dividing

Againk start with the positive whole numbers between 0 and 99 arranged in a grid. If a number is odd, multiply it by 3 and add 1. If a number is even, divide it by 2. Repeat the process using your answer as the next number. Repeat the process over and over again. What will happen in the long run? This time the black hole consists of a trio of numbers: 1, 2, and 4 (3 times 1 + 1 = 4, 4 divided by 2 = 2, and 2 divided by 2 equals 1), but it takes 115 iterations for every cell to get there!

0	4	1	1	2	2	2	1	4	1
4	2	4	4	2	2	1	4	2	2
1	1	4	4	1	2	1	4	4	4
4	1	2	2	1	1	4	4	4	1
2	1	2	2	1	1	1	2	2	4
4	4	2	2	1	1	1	2	1	2
1	1	2	2	4	4	4	4	2	2
2	4	1	1	1	2	1	1	2	2
4	1	2	2	4	4	4	4	2	4
2	2	2	2	4	4	4	8	1	1

Iteration 215

0	2	4	4	1	1	1	4	2	4
2	1	2	2	1	1	4	2	1	1
4	4	2	2	4	1	4	2	2	2
2	4	1	1	4	4	2	2	2	4
1	4	1	1	4	4	4	1	1	2
2	2	1	1	4	4	4	1	4	1
4	4	1	1	2	2	2	2	1	1
1	2	4	4	4	1	4	4	1	1
2	4	1	1	2	2	2	2	1	2
1	1	1	1	2	2	2	4	4	4

Iteration 216

0	1	2	2	4	4	4	2	1	2
1	4	1	1	4	4	2	1	4	4
2	2	1	1	2	4	2	1	1	1
1	2	4	4	2	2	1	1	1	2
4	2	4	4	2	2	2	4	4	1
1	1	4	4	2	2	2	4	2	4
2	2	4	4	1	1	1	1	4	4
4	1	2	2	2	4	2	2	4	4
1	2	4	4	1	1	1	1	4	1
4	4	4	4	1	1	1	2	2	2

Iteration 217

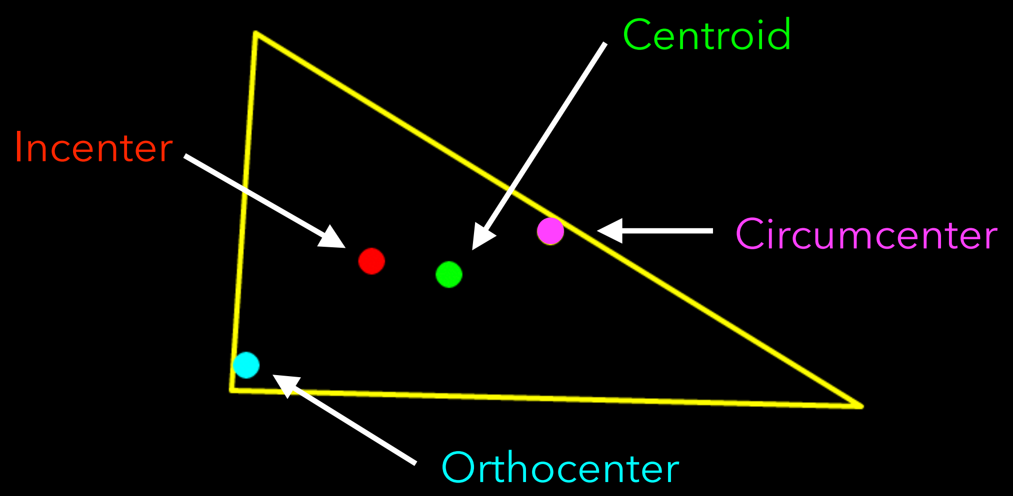
0	4	1	1	2	2	2	1	4	1
4	2	4	4	2	2	1	4	2	2
1	1	4	4	1	2	1	4	4	4
4	1	2	2	1	1	4	4	4	1
2	1	2	2	1	1	1	2	2	4
4	4	2	2	1	1	1	2	1	2
1	1	2	2	4	4	4	4	2	2
2	4	1	1	1	2	1	1	2	2
4	1	2	2	4	4	4	4	2	4
2	2	2	2	4	4	4	1	1	1

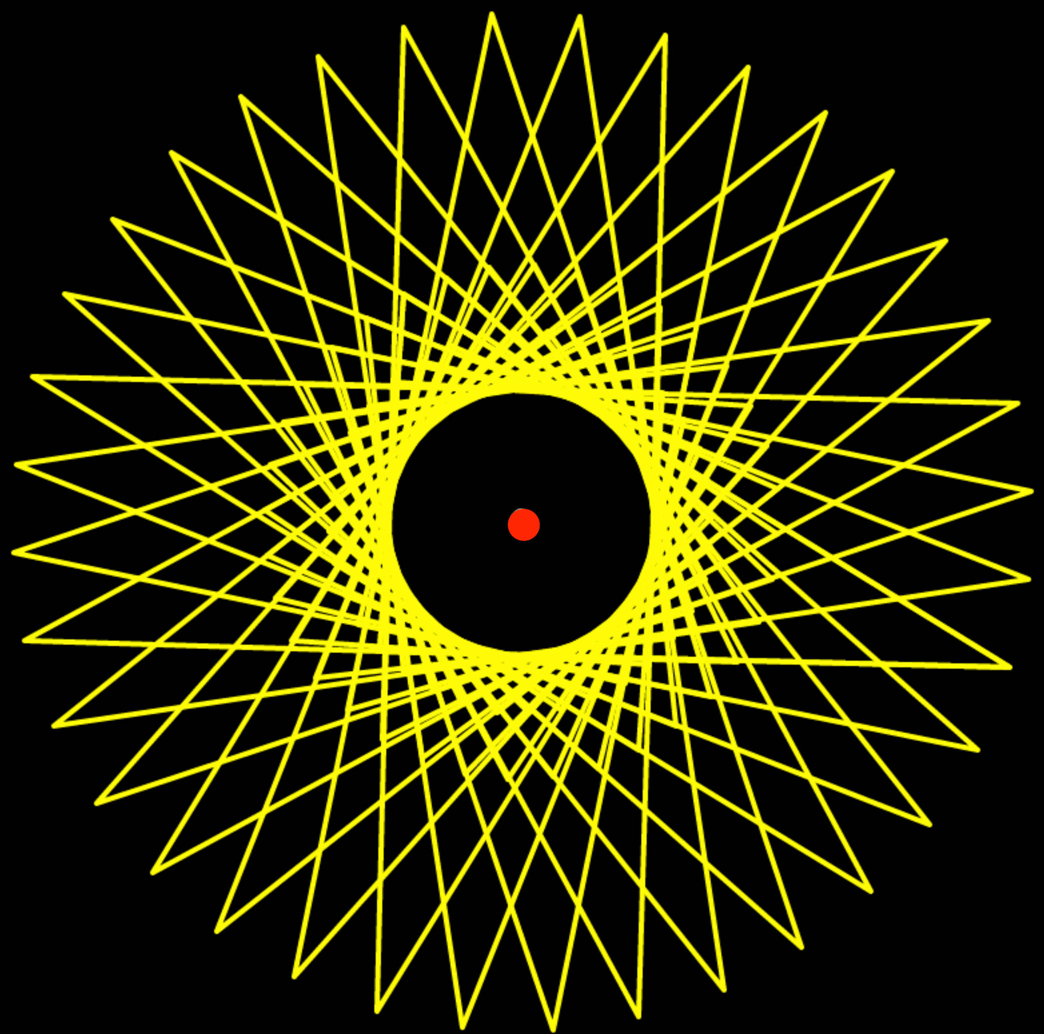
Iteration 218

Rotations and Revolutions

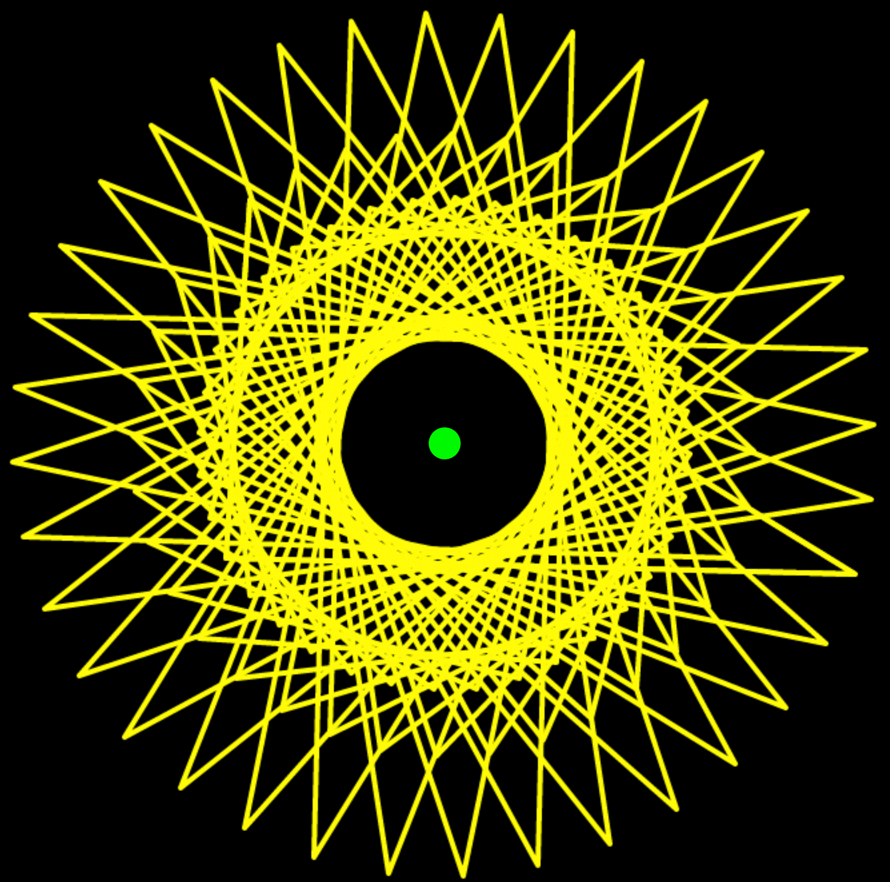
There are many defined "centers" of a triangle. Four of the best known are incenter, centroid, circumcenter, and orthocenter. Those centers are plotted for the triangle shown to the right.

What would look like if you rotated the triangle around each of those centers, saving a copy of the triangle every 10°?



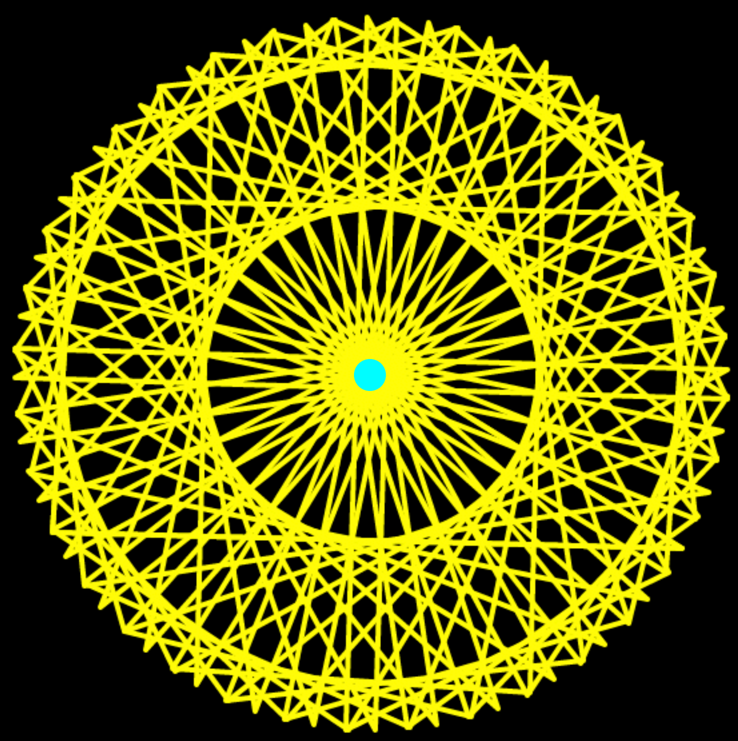


Incenter

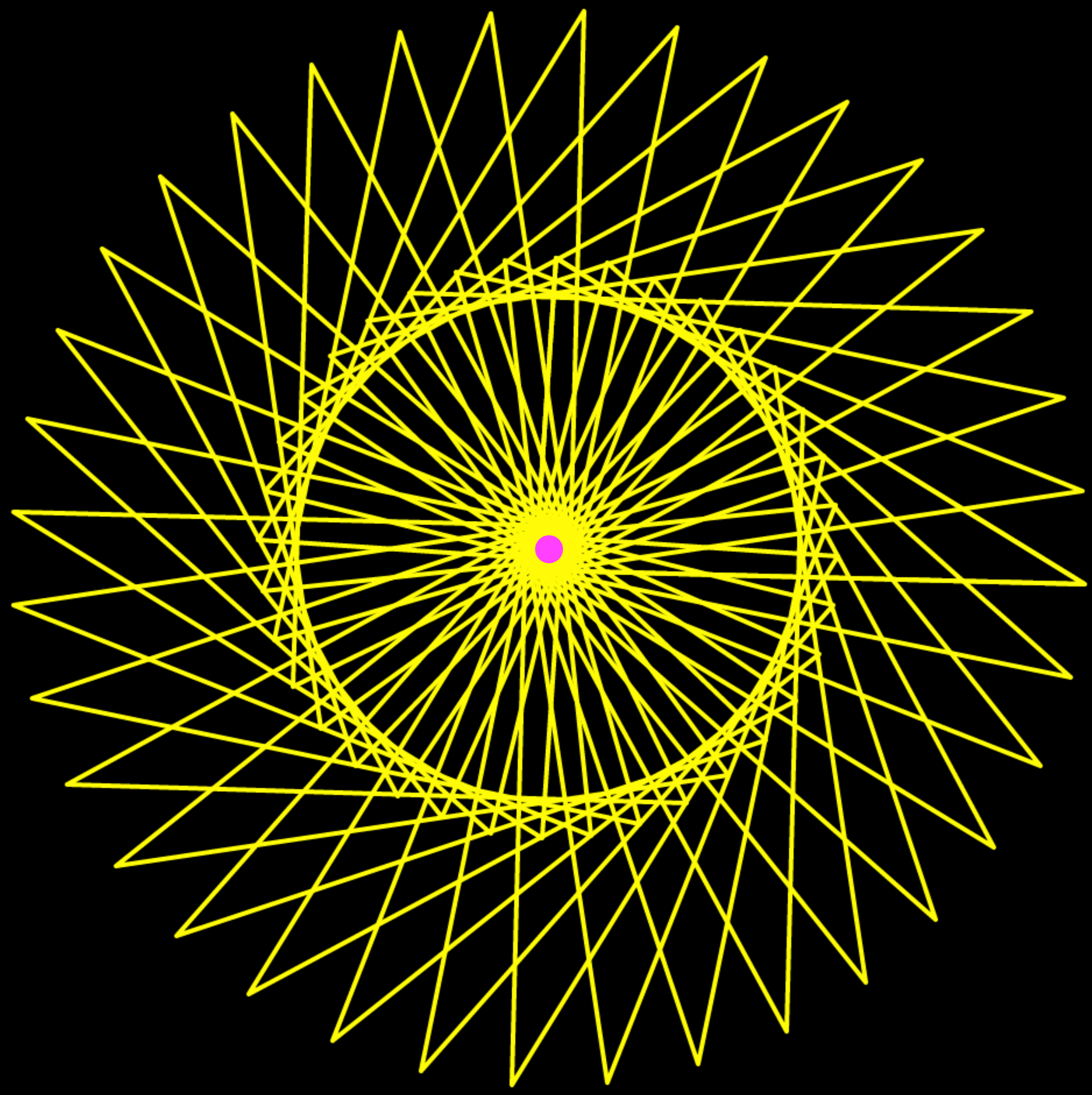


Centroid

More rotations around centers



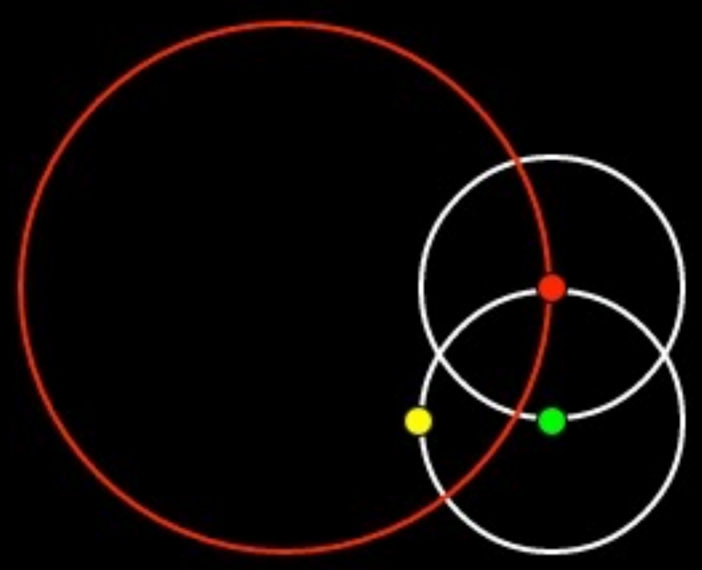
Orthocenter

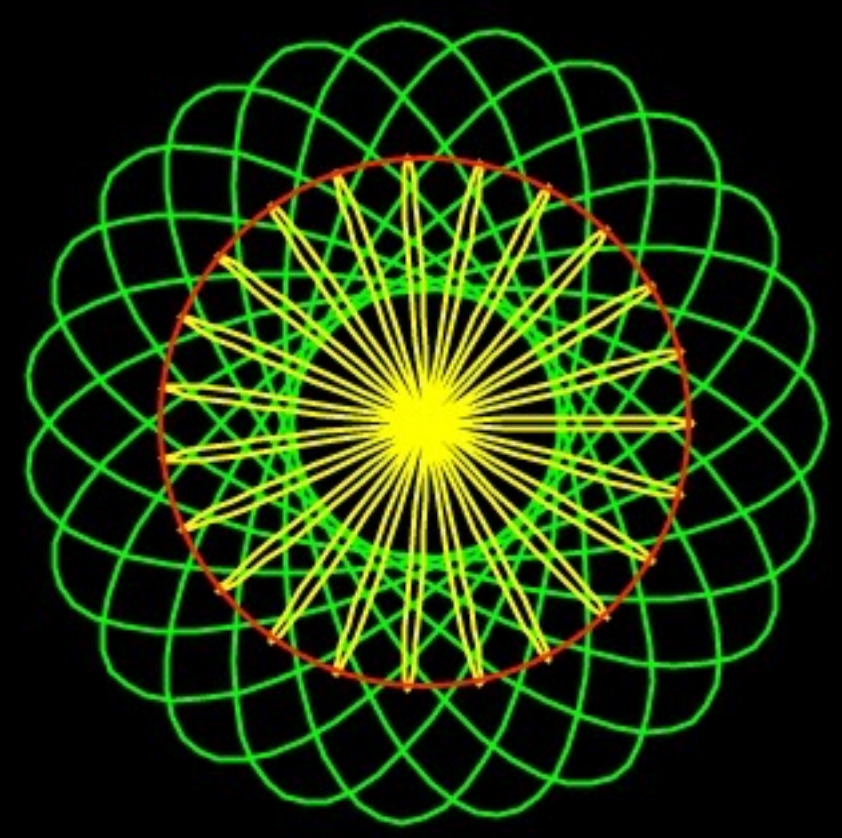


Circumcenter

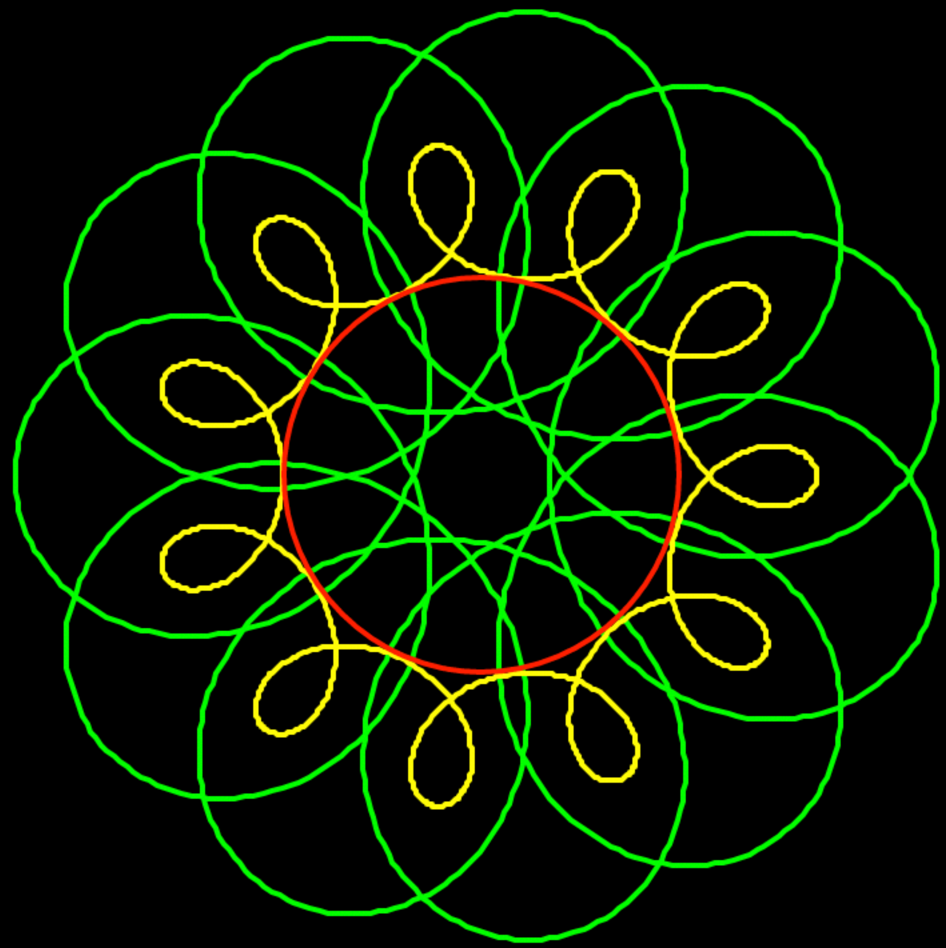
Objects revolving around revolving objects

The two smaller circles have half the diameter of the larger circle. Imagine the red dot revolving around the red circle, the green dot revolving around the red dot and the yellow dot revolving around the green dot. The red and yellow dots revolve at the same angular velocity in a clockwise direction, but he green dot revolves at 10/11 of that velocity in a counter-clockwise direction. As the dots move, their paths are traced.



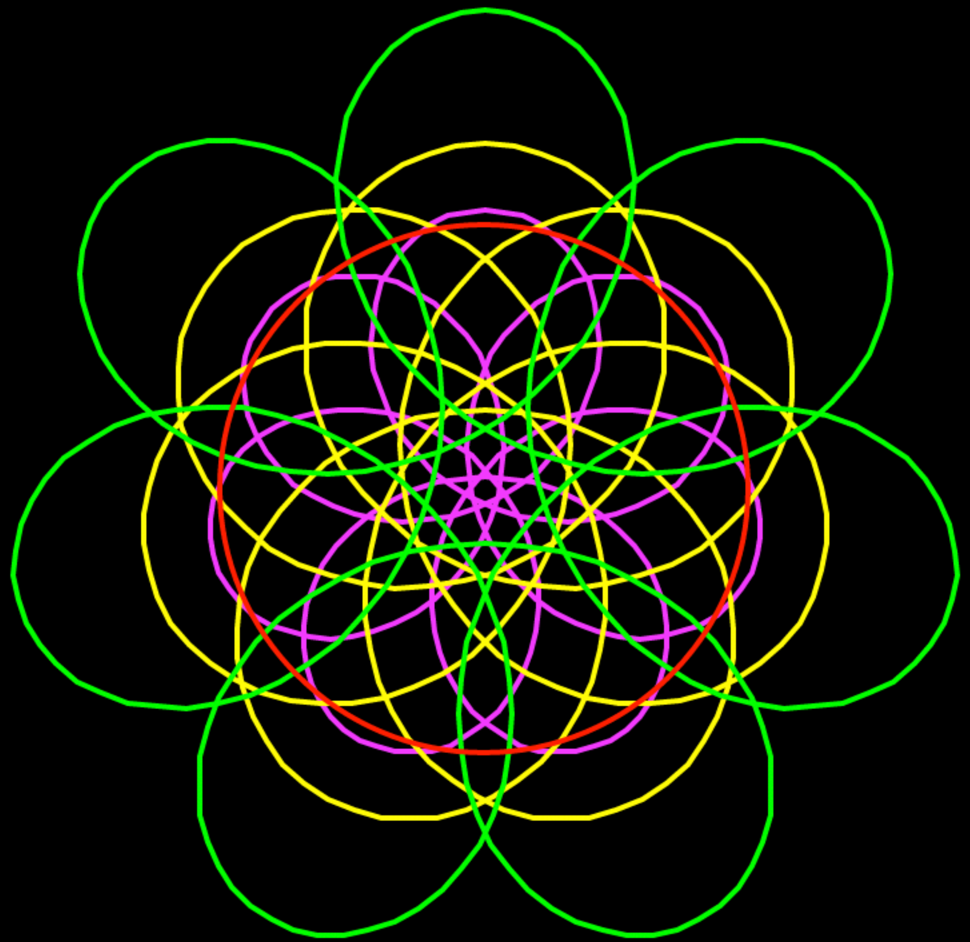


More objects revolving around revolving objects



Diameter ratios: 3, 4, 4

Angular velocity ratios: 10, -1, 10



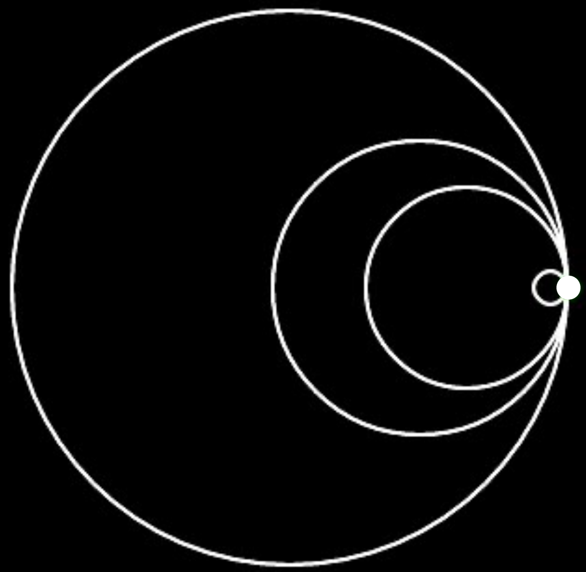
Diameter ratios: 4, 3.2, 2, 1

Angular velocity ratios: 1, -6, 1, -6

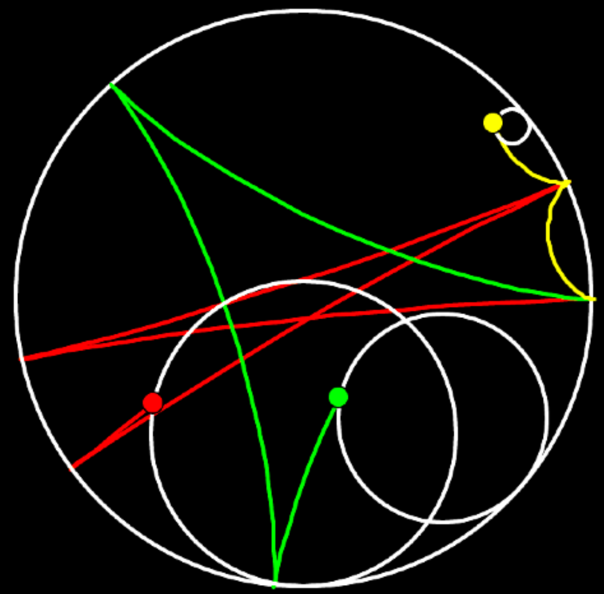
Circles rolling around circles

Three circles roll around the inside of a larger circle. A colored dot marks a spot on each rolling circle. The angular velocity of each dot, relative to the center of its circle, is set to be the same, so the circumference of each circle determines how quickly it moves around the given circle. A pattern is created by tracing the paths of the dots.

Initial position with all three dots at the point of tangency.

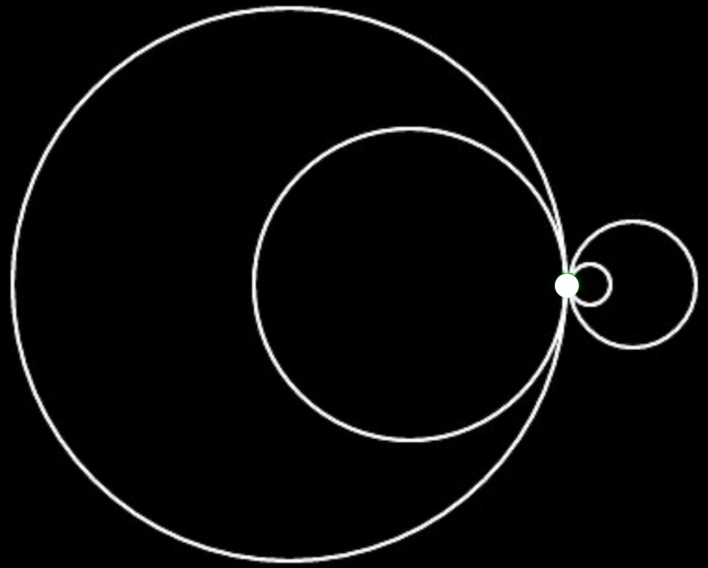


After each smaller circle has completed two full rotations.

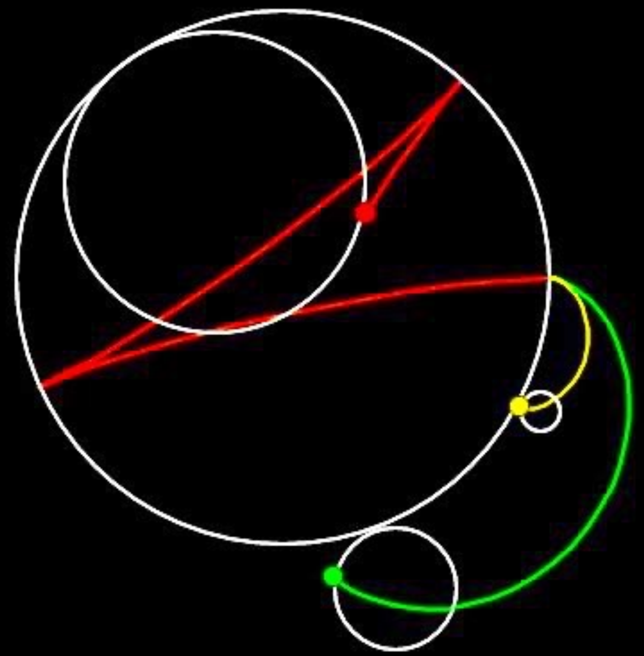


One circle rolls around the inside of a larger circle while two circles roll around the outside of the circle.

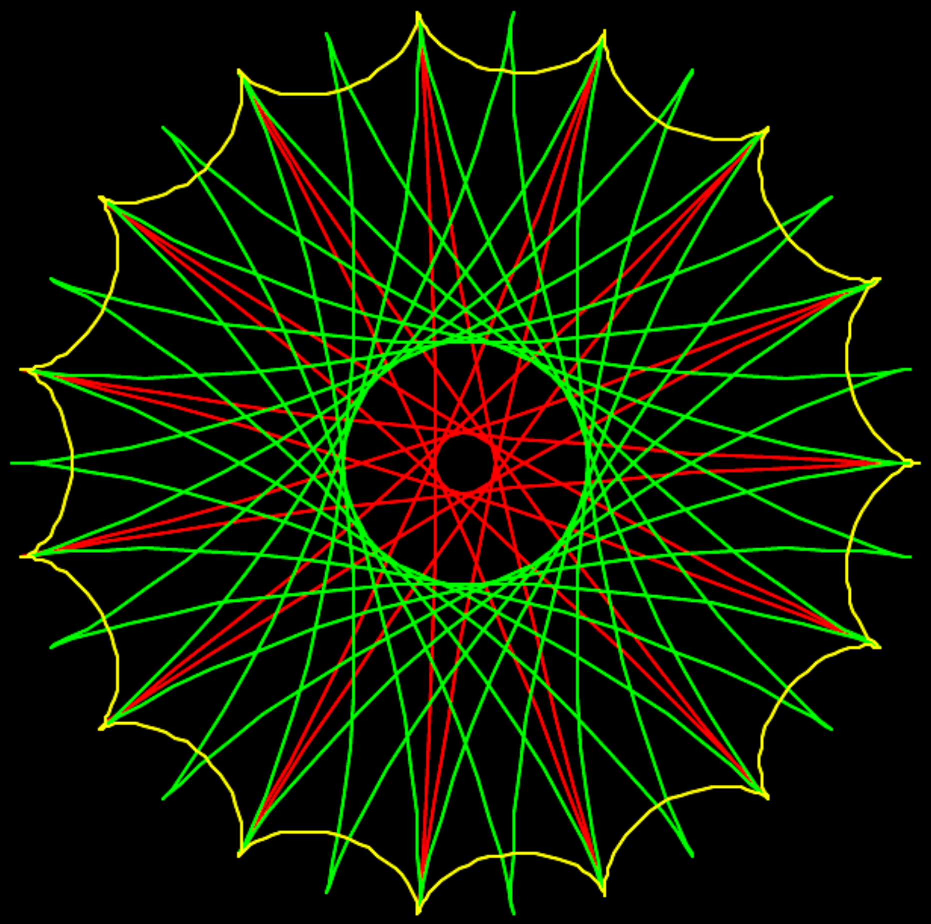
Initial position with all three dots at the point of tangency.



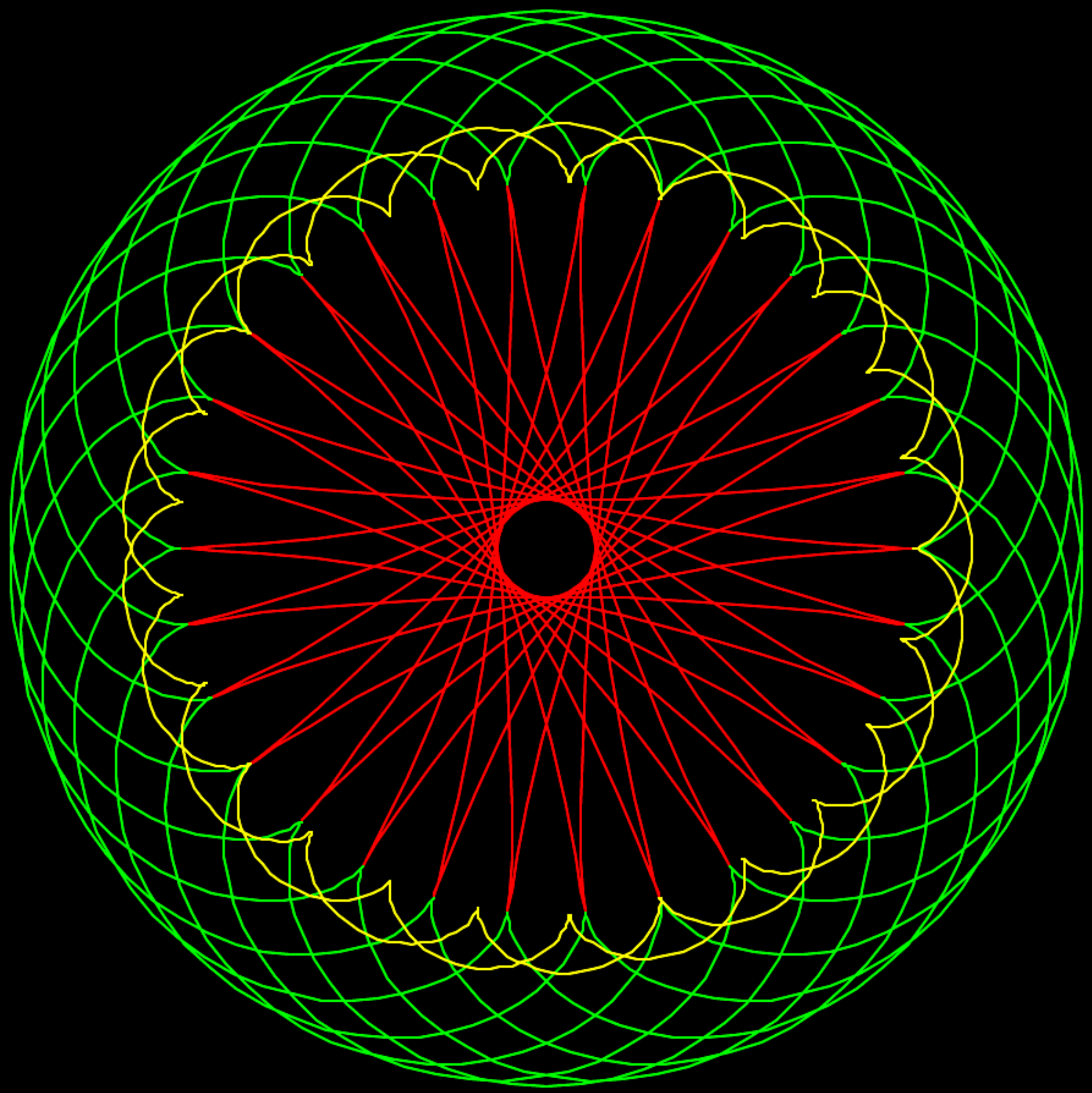
After each smaller circle has completed one full rotation.



Rolling circles after many rotations



The radius of the large circle is 150, and the radii of the rolling circles are 80, 55, and 10.



The radius of the large circle is 150, and the radii of the rolling circles are 85, 35, and 12.

The end of this book
but just the beginning of the search.

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